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„Műszaki és gazdasági szakok alapozó matematikai
ismereteinek e–learning alapú tananyag- és módszertani fejlesztése”

College Algebra

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Chapter 1

Real Numbers

1.1 Introduction of real numbers

The most important notion in mathematics is the notion of numbers. Numbers are classified and special sets of numbers are named based on several properties of the numbers. The most important set of numbers is the set of **real numbers**. A *real number* is a number which can be written as a *decimal*. Decimals might be *finite decimals* and *infinite decimals*, which are either *infinite repeating decimals* or *infinite non-repeating decimals*. Finite and infinite repeating decimals are also called **periodic decimals**, meanwhile infinite non-repeating decimals are called **non-periodic decimals**.

Example. Finite decimals are like

$$\frac{3}{2} = 1.5 \qquad \frac{1}{25} = 0.04$$

infinite repeating decimals are like

$$\frac{2}{3} = 0.6666\ldots \qquad \frac{1}{7} = 0.142857142857\ldots \qquad \frac{44}{175} = 0.25142857142857\ldots$$

and infinite non-repeating decimals are like

$$\sqrt{2} = 1.4142135623730\ldots \qquad \pi = 3.14159265358979323846264 3\ldots$$

On the set of real numbers there are defined several operations. The two most important operations are **addition** and **multiplication** of real numbers. In the sequel we assume that addition and multiplication of real numbers is well known for the reader.

Notation. The set of real numbers is denoted by $\mathbb{R}$.

1.2 Axiomatic definition of real numbers*

Definition 1.1. Let F be a non-empty set with two operations $+$ and $\cdot$ with the following properties

(A1) For all $a, b, c \in F$, $(a + b) + c = a + (b + c)$. (+ associative)

(A2) For all $a, b \in F$, $a + b = b + a$. (+ commutative)

(A3) There exists $0 \in F$ such that for all $a \in F$, $a + 0 = a$. (Zero)

(A4) For all $a \in F$, there exists $(-a) \in F$ such that $a + (-a) = 0$. (Negatives)

(M1) For all $a, b, c \in F$, $(ab)c = a(bc)$. ($\cdot$ associative)

(M2) For all $a, b \in F$, $ab = ba$. ($\cdot$ commutative)

(M3) There exists $1 \in F$, $1 \neq 0$, s.t. for all $a \in F$, $a1 = a$. (Unit)

(M4) For all $0 \neq a \in F$, there exists $(1/a) \in F$ such that $a\cdot(1/a) = 1$. (Reciprocals)

(DL) For all $a, b, c \in F$, $a(b + c) = ab + ac$. (Distributive)

Further, there exists a relation $\leq$ on F such that

(O1) For all $a, b \in F$, either $a \leq b$ or $b \leq a$.

(O2) For all $a, b \in F$, if $a \leq b$ and $b \leq a$, then $a = b$.

(O3) For all $a, b, c \in F$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

(O4) For all $a, b, c \in F$, if $a \leq b$, then $a + c \leq b + c$.

(O5) For all $a, b, c \in F$, if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Finally, every nonempty subset of F that has an upper bound also has a least upper bound (supremum). In this case $F = \mathbb{R}$.

More precisely it can be shown that the axioms above determine $\mathbb{R}$ completely, that is, any other mathematical object with the same properties must be essentially the same as $\mathbb{R}$.

1.3 Properties of real numbers

The below basic properties of addition and multiplication of real numbers are so-called axioms, so they are assumed to be self-evidently true and there is no need to prove them.

Definition 1.2. (Axioms of the Real Numbers)
Let a, b and c be arbitrary real numbers. Then the following properties are true:

Closure properties	$a + b$ is a real number
	ab is a real number
Commutative properties	$a + b = b + a$
	$ab = ba$
Associative properties	$(a + b) + c = a + (b + c)$
	$(ab)c = a(bc)$
Identity properties	There exists a unique real number 0 called *zero* such that $a + 0 = a \quad \text{and} \quad 0 + a = a$
	There exists a unique real number 1 called *one* such that $a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a$
Inverse properties	There exists a unique real number $-a$ called *the additive inverse* of a such that $a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0$
	If $a \neq 0$ then there exists a unique real number $\frac{1}{a}$ called *the multiplicative inverse* of a such that $a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1$
Distributive property	$a(b + c) = ab + ac$

Remark. The additive inverse of a is also called *the negative* of a or *the opposite* of a, and the multiplicative inverse of a is also called *the reciprocal* of a.

Remark. Do not confuse "the negative of a number" with "a negative number"!

Notation. The reciprocal of a is also denoted by a^{-1}.

Theorem 1.3. (Properties of the real numbers) *Let a, b and c be arbitrary real numbers. Then the following properties are true:*

Substitution property *If $a = b$ then a and b may replace each other in any expression.*

Addition property *If $a = b$ then*

$$a + c = b + c$$

Multiplication property *If $a = b$ then*

$$ac = bc$$

Theorem 1.4. (Properties of zero)
For all real numbers a and b we have

1). $a \cdot 0 = 0$,

2). $ab = 0$ *if and only if* $a = 0$ *or* $b = 0$.

Theorem 1.5. (Properties of the additive inverse) *For all real numbers a and b we have*

1). $-(a + b) = (-a) + (-b)$

2). $-(-a) = a$

3). $(-a)b = -(ab)$

4). $a(-b) = -(ab)$

5). $(-a)(-b) = ab$

There are other operations on real numbers which may be defined using the addition and multiplication. The subtraction of two real numbers is defined by the addition of the additive inverse of the second to the first. Similarly, the division of a real number by a non-zero real number is defined in terms of multiplication.

Definition 1.6. (Definition of subtraction and division) For all real numbers a and b we define the difference $a - b$ by

$$a - b = a + (-b)$$

Similarly, for all real numbers a and $b \neq 0$ we define the quotient $\frac{a}{b}$ by

$$\frac{a}{b} = a \cdot b^{-1}.$$

Notation. The quotient $\frac{a}{b}$ is also denoted by $a : b$.

Remark. Clearly, you may divide zero by any nonzero number, and the result is zero:

$$\frac{0}{3} = 0, \quad \frac{0}{-\pi} = 0, \quad \frac{0}{2\sqrt{3}} = 0.$$

On the other hand **dividing any number by zero is meaningless**.

Theorem 1.7. (Properties of subtraction and division of real numbers) *Let a, b, c, d be arbitrary real numbers, and suppose that all the denominators in the formulas below are non-zero:*

1). $0 - a = -a$

2). $a - 0 = a$

3). $-(a + b) = -a - b$

4). $-(a - b) = b - a$

5). $\frac{a}{b} = \frac{c}{d}$ *if and only if* $ad = bc$

6). $\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$

7). $\frac{-a}{-b} = \frac{a}{b}$

8). $\frac{ac}{bc} = \frac{a}{b}$

9). $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

10). $\frac{a}{b} : \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$

11). $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

12). $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Theorem 1.8. (Cancelation rule of addition) *Let a, b, c be arbitrary real numbers. If $a + c = b + c$, then $a = b$.*

Theorem 1.9. (Cancelation rule of multiplication) *Let a, b, c be arbitrary real numbers, with $c \neq 0$. If $ac = bc$, then $a = b$.*

Later we shall define two other operations of the real numbers: exponentiation and taking roots.

1.4 The order of operations and grouping symbols

If no parentheses and fraction lines are present then we first have to do all exponentiations and taking roots, then multiplications and divisions working from left to right, and then we have to do all the additions and subtractions working from left to right.

To change the order of the operations we use **grouping symbols**: *parentheses* ( ), *square brackets* [ ], and *braces* { }. Further, we remark that root symbols and fraction lines also work as grouping symbols.

Order of operations:

I. If no fraction lines and grouping symbols are present:

   (1) First do all exponentiations and taking roots in the order they appear, working from left to right,

   (2) Then do all multiplications and divisions in the order they appear, working from left to right.

   (3) Finally do all additions and subtractions in the order they appear, working from left to right.

II. If there are fraction lines and/or grouping symbols present:

   (1) Work separately above and below any fraction line, and below any root sign.

   (2) Use the rules of point I. within each parentheses, square brackets and braces (and any other grouping symbols) starting with the innermost and working outwards.

Example.

$$1 + 2 \cdot 3 = 1 + 6 = 7 \qquad \text{but} \qquad (1 + 2) \cdot 3 = 3 \cdot 3 = 9$$

Example.

$$\frac{7 + \sqrt{3 + 2 \cdot 3}}{(3 + \sqrt{4}) \cdot 2} = \frac{7 + \sqrt{3 + 6}}{(3 + 2) \cdot 2} = \frac{7 + \sqrt{9}}{5 \cdot 2} = \frac{7 + 3}{10} = \frac{10}{10} = 1$$

Remark. In mathematics we try to avoid the use of slash indicating division. However, if used, then it means just a sign of division (:), and not a fraction line. This means that

$$1/2 \cdot 4 = 1 : 2 \cdot 4 = 0.5 \cdot 4 = 2.$$

1.5 Special Subsets of the Set of Real Numbers

During the history of mathematics the real numbers were not the first set of numbers which appeared. It seems that the set of natural numbers $(1, 2, 3, \dots)$ developed together with the human race, being with us from the beginning. In contrast already zero, and the negative integers are recent inventions in mathematics. Partly, they are results of the "wish" to be able to subtract any two natural numbers. To be able to divide any integer by any non-zero integer the set of rational numbers has been introduced as the set of all quotients of integer numbers. However, these "do not fill completely" the coordinate line, so mathematicians introduced the set of real numbers, containing the set of rational numbers, and the set of irrational numbers, which is the set of all non-rational real numbers. Here we summarize the definition of these sets:

The set of **natural numbers** contains all numbers which can be obtained by successively adding several copies of 1:

$$\mathbb{N} := \{1, 2, 3, \dots\}$$

The set of **integer numbers** contains the natural numbers, their negatives and zero:

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The set of **rational numbers** contains all numbers which can be obtained by dividing an integer by a non-zero integer:

$$\mathbb{Q} := \left\{ \tfrac{a}{b} \,|a, b \in \mathbb{Z}, b \neq 0 \right\}$$

The set of **irrational numbers** contains all real numbers which are not rational numbers:

$$\mathbb{R} \setminus \mathbb{Q} := \{r \in \mathbb{R} | r \notin \mathbb{Q}\}$$

Definition 1.10. (Positive and negative real numbers) Zero is by definition neither positive nor negative. Real numbers which have their decimal form starting with a natural number (e.g. 3.141592....) or which start with zero (e.g. 0.0012....) are called **positive numbers**, and their additive inverses are called **negative numbers**.

Remark. (Properties of the sign of real numbers)

- Zero is neither positive nor negative.

- All natural numbers are positive.
- The additive inverse of a natural number is negative.
- A rational number is positive if
  - both the numerator and the denominator is positive
  - both the numerator and the denominator is negative
- A rational number is negative if
  - the numerator is negative and the denominator is positive
  - the numerator is positive and the denominator is negative

1.6 The Real Number Line and Ordering of the Real Numbers

The real number line is a geometric representation of the set of real numbers. In many cases this geometric representation helps us to understand the structure of the set of real numbers. This is the case especially with the ordering of real numbers.

Definition 1.11. (The Real Number Line) Take a straight line (for simplicity draw it horizontally) and fix any point on the line to represent 0. This point will also be called *the origin*. Then choose any point on the right of this point and label it by 1. This way using the distance of these two points we have fixed a unit measure. With the help of this we can locate $2, 3, 4 \dots$ to the right of the origin and using central symmetry through the origin we also fix the negatives of the natural numbers (i.e. $-1, -2, -3, \dots$) on the left of the origin. Dividing the segments between two consecutive integers we also locate the rational numbers which are not integers (like $\frac{1}{4}, -\frac{3}{5}, \frac{111}{31}$). Irrational numbers can be located by computing their decimal representation to any desired accuracy.

The number corresponding to a point is called the *coordinate* of the point, and the correspondence between the points on a line and the real numbers is called a *coordinate system.*

Now we define a "natural" ordering among the real numbers. There are may ways to define this ordering. The easiest way is to say that the real number a is larger then b if a is to the right of b on the coordinate line. Another way to define this ordering is the following: first define the concept of positive and negative numbers, then use this to define the above mentioned ordering.

Definition 1.12. Let a, b be real numbers. We say that a is greater than b, and write $a > b$, if $a - b$ is positive. Further, we say that a is smaller than b, and write $a < b$, if $a - b$ is negative.

Remark. We use several variations of the relations $>$ and $<$. The relation $\leq$ means smaller or equal, and $\geq$ stands for greater or equal, and for negation of statements involving such symbols we use the notations $\not>$, $\not<$, $\not\geq$, $\not\leq$.

Theorem 1.13. (Properties of the strict ordering) *Let a, b and c be arbitrary real numbers. Then the following properties of the ordering $<$ ("strictly smaller") are true:*

Irreflexive property	*The statement* $a < a$ *is always false.*
Strict antisymmetry property	*The statements* $a < b$ *and* $b < a$ *are never true simultaneously*
Transitive property	*If* $a < b$ *and* $b < c$ *then* $a < c$
Addition property	*If* $a < b$ *then* $a + c < b + c$
Multiplication property	*If* $a < b$ *and* $c > 0$ *then* $ac < bc$
	If $a < b$ *and* $c < 0$ *then* $ac > bc$
Trichotomy property	*For two given real numbers* a *and* b *one of the following three statements is always true* $a < b$, $b < a$ *or* $a = b$

Remark. The ordering $>$ ("strictly greater") has completely similar properties.

Theorem 1.14. (Properties of the non-strict ordering) *Let* a, b *and* c *be arbitrary real numbers. Then the following properties of the ordering* $\leq$ *("smaller or equal") are true:*

Reflexive property	*The statement* $a \leq a$ *is always true.*
Antisymmetry property	*If* $a \leq b$ *and* $b \leq a$ *then we have* $a = b$
Transitive property	*If* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$
Addition property	*If* $a \leq b$ *then* $a + c \leq b + c$
Multiplication property	*If* $a \leq b$ *and* $c \geq 0$ *then* $ac \leq bc$
	If $a \leq b$ *and* $c \leq 0$ *then* $ac \geq bc$
Dichotomy property	*For two given real numbers* a *and* b *one of the following two statements is always true* $a \leq b$ *or* $b \leq a$

Remark. The ordering $\geq$ ("greater or equal") has completely similar properties.

1.7 Intervals

Intervals are the sets corresponding to segments or semi-lines of the coordinate line. An interval is a set containing all real numbers between the two endpoints of the interval. In the interval notation we use square brackets around the two endpoints of the interval, and the direction of the square bracket also indicates if the endpoint is included in the set or not.

Definition 1.15. Let $a \leq b$ be real numbers. Then we define the following types of intervals:

- **Open intervals**:

$$]-\infty, b[:= \{x \in \mathbb{R} \mid x < b\}$$

$$]a, b[:= \{x \in \mathbb{R} \mid a < x < b\}$$

$$]a, \infty[:= \{x \in \mathbb{R} \mid a < x\}$$

- **Half-open intervals**:

$$]-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$$

$$[a, b[:= \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$]a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, \infty[:= \{x \in \mathbb{R} \mid a \leq x\}$$

- **Closed interval**

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Remark. If $a = b$ then we have the following conventions:

$$[a, a] = \{a\} \qquad \text{and} \qquad]a, a[= [a, a[=]a, a] = \emptyset$$

In principal it is also possible to define intervals with the left endpoint larger then the right endpoint such intervals representing always the empty set. This can be useful when writing down proofs including intervals where the endpoints are unknowns (i.e. letters) so a priory we do not know which of them is smaller or larger, but we will never write down an interval with given numbers as endpoints so that the left endpoint is larger than the right one.

Remark. If the square bracket "is looking toward the center" of the interval then the endpoint is included, otherwise it is not included in the set. So the same kind of bracket has different meaning at the left and right endpoint of the interval. This can be explained in the following way:

- the sign] **at the left** endpoint of an interval means that
  the endpoint is not included in the set
- the sign] **at the right** endpoint of an interval means that
  the endpoint is included in the set
- the sign [**at the left** endpoint of an interval means that
  the endpoint is included in the set
- the sign [**at the right** endpoint of an interval means that
  the endpoint is not included in the set

Exercise 1.1. Decide which of the real numbers $-7, -5.3, -5, -4.99, -\pi, -1, 0, 1,$ $\sqrt{3}$, 3.99, $4, 4.02, 5$ are included in the interval:

a) $]-\infty, -5[$	b) $]-\infty, -5]$	c) $]-5, 4[$
d) $[-5, 4[$	e) $]-5, 4]$	f) $[-5, 4]$
g) $[-5, \infty[$	h) $]-5, \infty[$	f) $\left[\sqrt{3}, 5\right[$

Draw the graph representing the above intervals on a real number line.

Exercise 1.2. Determine the intervals containing all real numbers fulfilling the following condition

a) $2 < x < 7$	b) $x \geq 3$	c) $x \leq 3$
d) $5 \leq x < 8$	e) $5 < x \leq 11$	f) $x < 7$
g) $-1 < x$	h) $x > -1$	f) $1 \leq x \leq 3$

1.8 The Absolute Value of a Real Number

Definition 1.16. Let x be a real number. The **absolute value** of x is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.1}$$

Remark. The absolute value of a real number is in fact the distance from the origin of the point which represents that number on the number line.

Equivalent definitions for the absolute value:

All formulas below are equivalent reformulations of (1.1), so they are equivalent definitions of the absolute value:

$$|x| := \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases} \qquad |x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases} \qquad |x| := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem 1.17. (Properties of absolute value)
Let $a, b \in \mathbb{R}$ be arbitrary real numbers. Then we have

- $|a| \geq 0$,
- $|-a| = |a|$,
- $|a \cdot b| = |a| \cdot |b|$,
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$,
- $|a+b| \leq |a| + |b|$,
- $|a| = b$ *if and only if* $a = b$ *or* $a = -b$,
- $|a| < b$ *if and only if* $-b < a < b$,
- $|a| > b$ *if and only if* $a < -b$ *or* $a > b$.

1.8.1 The graphical approach of absolute value function

We consider an example. Let $f(x) = 2|x-5| + 4$ and give the graph of this function. First take the graph of the function $|x|$, see figure 1.1.

Next figure 1.2 shows the function $|x-5|$.

In the third step we have the graph of $2|x-5|$, see figure 1.3.

Finally, the graph of $f(x)$ is on figure 1.4.

We remark that this geometrical approach is useful to solve certain equations containing absolute value(s).

1.9 Exponentiation

1.9.1 Integer exponents

Let a be a real number and n a natural number. To shorten the notation for the repeated multiplication $a \cdot a \cdot \ldots \cdot a$ (a appearing n times) we introduce the

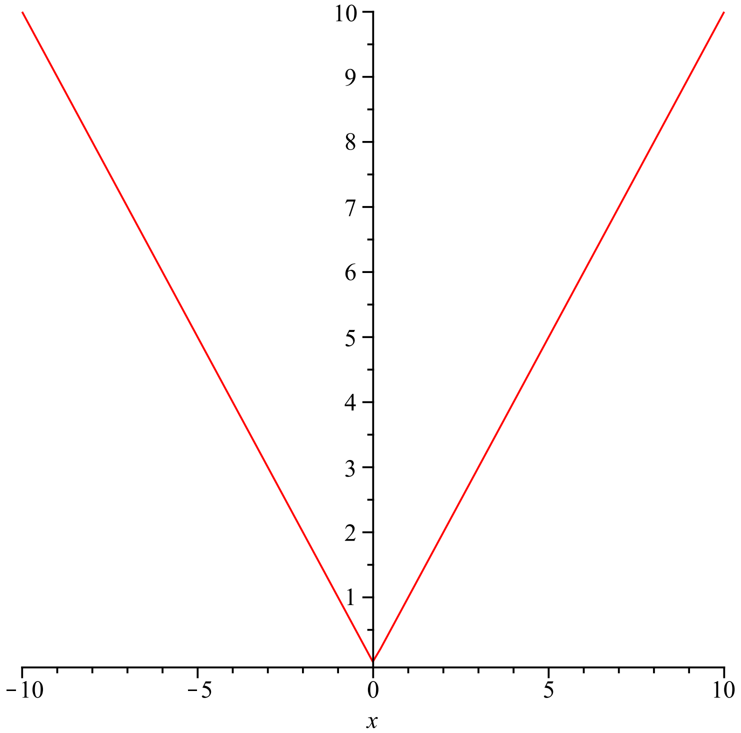

Figure 1.1: Graph of the function $|x|$

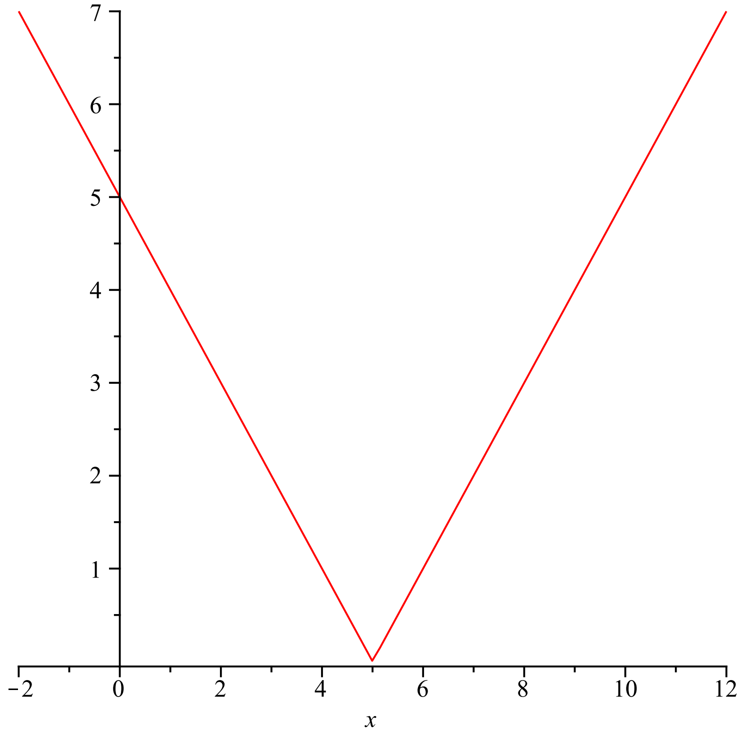

Figure 1.2: Graph of the function $|x - 5|$

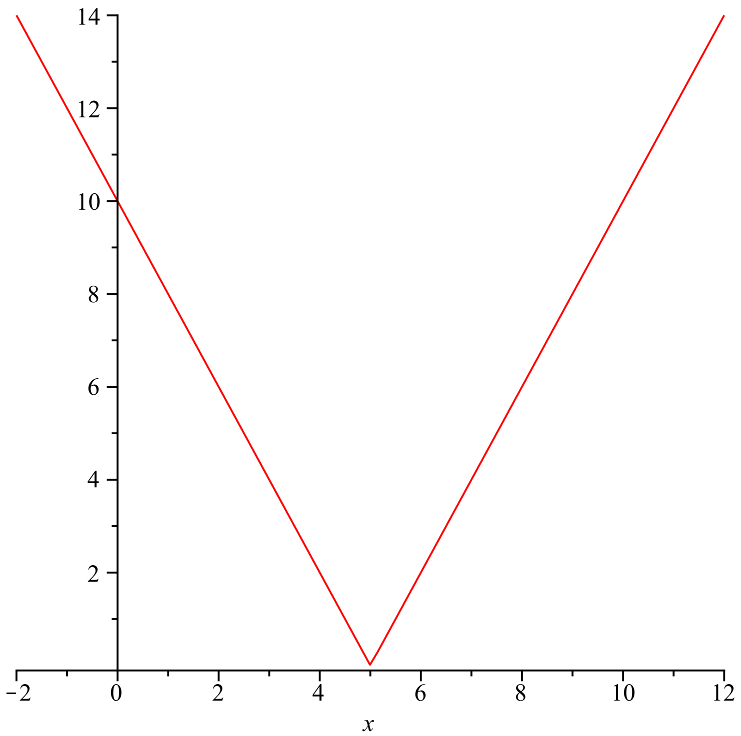

Figure 1.3: Graph of the function $2|x-5|$

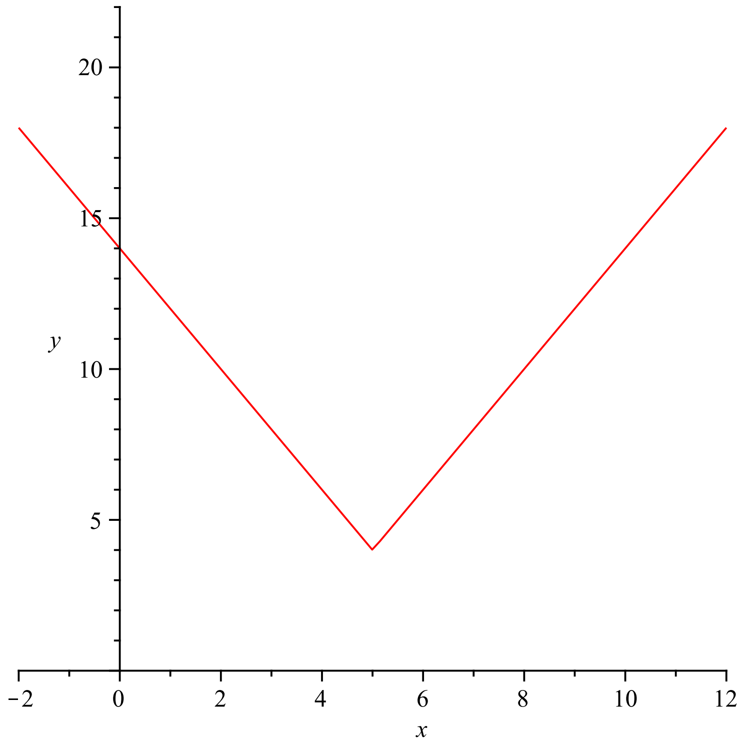

Figure 1.4: Graph of the function $2|x-5|+4$

exponential notation a^n, i.e.

$$a^n := \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}.$$

Here we call a the **base** and n the **exponent** or the **power**. Further, put $a^0 := 1$ and for $a \neq 0$ put $a^{-n} := \frac{1}{a^n}$ for any natural number n. This way we have defined the exponentiation for any non-zero real base and any integer exponent.

Theorem 1.18. (Properties of exponentiation with integral exponents) *Let m, n be integers and a, b real numbers. Further, if a or b is zero then suppose that m and n are positive. Then we have*

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example. Simplify the following expression containing exponentiations:

$$\frac{(a^3)^2 \cdot a^4 \cdot (a^2)^5}{a^7 \cdot (a^2)^4}; \qquad a \neq 0.$$

Solution:

$$\frac{(a^3)^2 \cdot a^4 \cdot (a^2)^5}{a^7 \cdot (a^2)^4} = \frac{a^6 \cdot a^4 \cdot a^{10}}{a^7 \cdot a^8} = \frac{a^{6+4+10}}{a^{7+8}} = \frac{a^{20}}{a^{15}} = a^{20-15} = a^5.$$

Example. Simplify the following expression containing exponentiations:

$$(a^4 b^{-2})^{-2} \cdot \frac{a^6 b}{a^{-2} b^5}; \qquad a \neq 0,\ b \neq 0.$$

Solution:

$$(a^4 b^{-2})^{-2} \cdot \frac{a^6 b}{a^{-2} b^5} = (a^4)^{-2} \cdot (b^{-2})^{-2} \cdot a^{6-(-2)} b^{1-5} = a^{-8} \cdot b^4 \cdot a^8 b^{-4} = a^{-8+8} \cdot b^{4+(-4)} = a^0 b^0 = 1.$$

Exercise 1.3. Simplify the following expression containing exponentiations:

a) $2^6 \cdot 2^7$	b) $\dfrac{2^6}{2^4}$	c) $10^6 \cdot \left(\dfrac{1}{5}\right)^6$
d) $(2^4)^3$	e) $\dfrac{8^4}{4^4}$	f) $\dfrac{a^7 \cdot a^{-4}}{a^3}$
g) $3^6 \cdot 3^3 : 3^7$	h) 3^{2^5}	i) $(3^2)^5$
j) $\dfrac{(2^5 \cdot 3^4 \cdot 5^3) \cdot (2^4 \cdot 3^3 \cdot 5^2 \cdot 7^3)}{2^6 \cdot 3^5 \cdot 5^5 \cdot 7}$	k) $2^{4^2} : (2^4)^3$	l) $(-3a^2b^4c^3)^2$
m) $4a^2b^3c^7 \cdot (-3ab^2c^5)$	n) $\dfrac{-27a^8b^9c^7}{-3a^4b^8c^7}$	o) $\dfrac{2^{-3}a^4b^{-5}c^{-2}}{3^{-2}a^{-2}b^{-3}c^{-1}}$
p) $\dfrac{2}{3}(a^3)^2b^{-3}c^4\left[-\dfrac{1}{2}a^4b^7c^{-3}\right]$	q) $\dfrac{5a^0b^{-3}(c^{-2})^0}{2^{-1}a^{-3}b^{-5}d^3}$	r) $\dfrac{a^3b^{-5}c^3}{a^{-2}b^{-7}c}$
s) $(-2x^2y^{-3}z^{-1})^{-3} \cdot (-1x^{-1}y^4z^5)^2$	t) $3x^2(y^3)^2z^4 \cdot 2(x^3y^2z)^3$	u) $\dfrac{a^{4^2}(b^3)^2}{a^{2^3}b^{2^2}}$

1.9.2 Radicals

Definition 1.19. Let a, b be real numbers, n a natural number. Suppose that if n is even, then a and b are positive. Then the n**th root of** a is denoted by $\sqrt[n]{a}$ and is defined by

$$\sqrt[n]{a} = b \quad \textbf{if and only if} \quad a = b^n.$$

For $\sqrt[2]{a}$ we use the notation $\sqrt{a}$.

Example.

$\sqrt{4} = 2, \quad \sqrt[3]{125} = 5, \quad \sqrt[5]{-32} = -2, \quad \sqrt[4]{-16}$ has no sense (it is not a real number)

Theorem 1.20. (Properties of radicals) *If n, k are positive integers and a, b are positive real numbers, then we have*

1). $\sqrt[n]{a^n} = \left(\sqrt[n]{a}\right)^n = a$

2). $\sqrt[k]{a^n} = \left(\sqrt[k]{a}\right)^n$

3). $\sqrt[n]{a}\,\sqrt[k]{a} = \sqrt[nk]{a^{n+k}}$

4). $\frac{\sqrt[n]{a}}{\sqrt[k]{a}} = \sqrt[nk]{a^{n-k}}$

5). $\sqrt[k]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}$

6). $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

7). $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Remark. The above statements are not necessarily true for the case when a, b might be negative. In one hand, for even n some of the expressions above have no sense in the set of real numbers. On the other hand, some of the above statements may be modified for the case of negative values of a, b.

In the case of statement 1) of Theorem 1.20 for even n and negative a we would have $\sqrt[n]{a^n} = |a| = -a$, while $\left(\sqrt[n]{a}\right)^n$ has no sense over the real numbers.

Example. Write the following expression using only one root sign:

$$\sqrt{a\sqrt[3]{a^2\sqrt[4]{a^3}}}; \qquad a \geq 0.$$

Solution:

$$\sqrt{a\sqrt[3]{a^2\sqrt[4]{a^3}}} = \sqrt{\sqrt[3]{a^3 \cdot a^2\sqrt[4]{a^3}}} = \sqrt[6]{a^5\sqrt[4]{a^3}} = \sqrt[6]{\sqrt[4]{(a^5)^4 \cdot a^3}} = \sqrt[24]{a^{20} \cdot a^3} = \sqrt[24]{a^{23}}$$

Example. Write the following expression using only one root sign:

$$\sqrt{a} \cdot \sqrt[3]{a^2} \cdot \sqrt[4]{a^5}; \qquad a \geq 0.$$

Solution:

$$\sqrt{a} \cdot \sqrt[3]{a^2} \cdot \sqrt[4]{a^5} = \sqrt[12]{a^6} \cdot \sqrt[12]{(a^2)^4} \cdot \sqrt[12]{(a^5)^3} = \sqrt[12]{a^6} \cdot \sqrt[12]{a^8} \cdot \sqrt[12]{a^{15}}$$
$$= \sqrt[12]{a^6 \cdot a^8 \cdot a^{15}} = \sqrt[12]{a^{6+8+15}} = \sqrt[12]{a^{29}}$$

Exercise 1.4. Simplify the following expression containing exponentiations, where all the indeterminates are supposed to be positive:

a) $\sqrt[3]{\sqrt{2}}$ b) $\sqrt{\sqrt[5]{3}}$ c) $\sqrt{\sqrt[3]{a^2}}$

d) $\sqrt[3]{\sqrt[4]{a^2 b}}$ e) $\sqrt{a\sqrt{a\sqrt{a}}}$ f) $\sqrt{a\sqrt{a^2\sqrt[4]{a^3}}}$

g) $\sqrt{\frac{x}{y}\sqrt{\frac{y}{x}\sqrt{\frac{x}{y}}}}$ h) $\sqrt[3]{a}\cdot\sqrt{b}\cdot\sqrt[4]{ab}$ i) $\sqrt{\frac{a}{b}}\cdot\sqrt[3]{\frac{b}{a}}\cdot\sqrt[6]{a}$

j) $\sqrt[3]{xy}\cdot\sqrt[5]{\frac{x}{y}}\cdot\sqrt[10]{\frac{y}{x}}$ k) $\sqrt{\frac{x}{y}\sqrt[3]{\frac{y}{x}}\sqrt[4]{xy}}$ l) $\sqrt[5]{x\sqrt[4]{\frac{1}{x}\sqrt[3]{x}}}$

m) $\sqrt{a\sqrt[3]{a}\sqrt[4]{a}}$ n) $\sqrt{a^3}\cdot\sqrt[3]{a^2}\cdot\sqrt[6]{a^{11}}$ o) $\sqrt[5]{a\sqrt[4]{a}}$

p) $\sqrt[3]{a^2\sqrt{a^3\sqrt{a}}}$ q) $\sqrt[7]{a}\cdot\sqrt[5]{a}\cdot\sqrt[3]{a}$ r) $\sqrt[4]{a^5\sqrt[3]{a^2\sqrt{a}}}$

1.9.3 Rational exponents

To extend the exponentiation for rational exponents for integers m, n $(n > 1)$ we put

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}. \tag{1.2}$$

However, this has no sense among the real numbers when $a < 0$ and n is even. Further, the laws of exponentiation with integer coefficient do not always generalize for the case of non-positive bases. For example, $((-1)^2)^{1/2} = 1$, but this is not equal to $(-1)^{2\cdot 1/2} = -1$. Thus in the sequel, whenever we use a non-integral rational exponent we have to restrict ourselves to positive bases.

Theorem 1.21. (Properties of exponentiation with rational exponents) *Let r, s be rational numbers, and let a, b be positive real numbers. Then we have*

1). $a^r \cdot a^s = a^{r+s}$

2). $\frac{a^r}{a^s} = a^{r-s}$

3). $a^{-r} = \frac{1}{a^r}$

4). $(a^r)^s = a^{rs}$

5). $(ab)^r = a^r b^r$

6). $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

Example. Simplify the following expression:

$$\left(a^{\frac{3}{4}}b^{\frac{1}{3}}\right)^{10} \cdot (\sqrt{a^3})^{-3} \cdot \sqrt[3]{b^2}; \qquad a > 0, b > 0.$$

Solution:

$$\left(a^{\frac{3}{4}}b^{\frac{1}{3}}\right)^{10} \cdot (\sqrt{a^3})^{-3} \cdot \sqrt[3]{b^2} = \left(a^{\frac{3}{4}}\right)^{10} \cdot \left(b^{\frac{1}{3}}\right)^{10} \cdot \sqrt{(a^3)^{-3}} \cdot \sqrt[3]{b^2}$$
$$= a^{\frac{3}{4}\cdot 10} \cdot b^{\frac{1}{3}\cdot 10} \cdot \sqrt{a^{-9}} \cdot \sqrt[3]{b^2} = a^{\frac{15}{2}} \cdot b^{\frac{10}{3}} \cdot a^{\frac{-9}{2}} \cdot b^{\frac{2}{3}}$$
$$= a^{\frac{15}{2}+\frac{-9}{2}} \cdot b^{\frac{10}{3}+\frac{2}{3}} = a^{\frac{6}{2}} \cdot b^{\frac{12}{3}} = a^3 b^4.$$

Exercise 1.5.

a) $\dfrac{\left(a^{\frac{2}{3}} \cdot b^{\frac{1}{2}}\right)^6}{a^3 \cdot b^2}$ b) $\left(a^{\frac{2}{3}} \cdot b^{\frac{1}{2}}\right)^{-2} \cdot a^{\frac{1}{3}} \cdot b^2$ c) $\left(a^{\frac{3}{2}} \cdot b^{\frac{5}{4}}\right)^{-2} \cdot a^5 \cdot b^3$

d) $\dfrac{\left(a^{\frac{4}{3}} \cdot b^{-\frac{1}{9}}\right)^9}{a^{10} \cdot b^2}$ e) $\dfrac{\left(a^{\frac{2}{5}} \cdot b^{\frac{1}{2}}\right)^{10}}{a^3 \cdot b^4}$ f) $\dfrac{\left(a^{\frac{1}{3}} \cdot b^{\frac{2}{5}}\right)^3}{a^3 \cdot b^2}$

g) $\dfrac{\left(a^{\frac{4}{3}} \cdot b^{\frac{3}{2}}\right)^5}{a^3 \cdot b^2}$ h) $\left(a^{\frac{3}{2}} \cdot b^{\frac{1}{2}}\right)^{-2} \cdot a^{\frac{5}{3}} \cdot b^2$ i) $\left(a^{\frac{2}{7}} \cdot b^{\frac{3}{5}}\right)^{-3} \cdot a^{\frac{14}{3}} \cdot b^{\frac{7}{3}}$

j) $\dfrac{\left(a^{-\frac{2}{3}} \cdot b^{\frac{1}{4}}\right)^6}{a^{-4} \cdot b^{\frac{3}{2}}}$ k) $\dfrac{\left(a^{-\frac{2}{5}} \cdot b^{\frac{3}{4}}\right)^6}{a^{-3} \cdot b^{\frac{3}{4}}}$ l) $\dfrac{\left(a^{-\frac{2}{7}} \cdot b^{\frac{3}{7}}\right)^{14}}{a^{-4} \cdot b^6}$

Chapter 2

Algebraic expressions

2.1 Introduction to algebraic expressions

In algebra it is common to use letters to represent numbers. If a letter may represent several numbers, then it is called a *variable*, if it represents a fixed value (like $\Pi = 3.141592\ldots$) then it is called a *constant*.

In mathematics variables are used in two ways. In one hand, there are variables, which represent a particular number (or some particular numbers) which have not yet been identified, but which have to be found. (An example for this use is the case of equations.) Such variables are also called **unknown**s. A second use of the variables is to describe general relationship between numbers, operations and other mathematical objects. (An example for this is the use of variables in describing axioms of real numbers.)

Definition 2.1. (Algebraic expression) An **algebraic expression** is the result of performing a finite number of the basic operations addition, subtraction, multiplication, division (except by zero), extraction of roots on a finite set of variables and numbers, and use of a finite number of grouping symbols. By **equivalent expressions** we mean expressions which represent the same real number for all valid replacements of the variables.

Remark. In many cases our goal is to simplify a given algebraic expression such that the result is an equivalent expression to the original one, which is much simpler in form. In the process of this simplification one may use only a restricted number of transformations, namely those which preserve the equivalence of the expressions. These are mainly the transformations described in our theorems.

Example.

$$\frac{a^{2/3} - ab^2}{a^{-1/3} \cdot \sqrt[5]{b}}, \quad x^2 - 3, \quad \frac{x^3 - 1}{x^2 + x + 1}$$

Definition 2.2. Here we define some basic notions connected to algebraic expr-essions:

- A **term** is the product of a real number and powers of one or more variables. The above-mentioned real number is called the **coefficient**.
- Two terms with the same variables rised to the same powers are called "**like terms**" or "**terms of the same type**". Like terms which are added or subtracted may be "collected" (using the distributive law) to get again a "like term" to the original ones whose coefficient can be computed by adding or subtracting the coefficients of the original "like terms".
- A **monomial** is a term in which all variables are raised to non-negative integer powers.
- The **degree of a monomial** is the sum of the exponents of all unknowns in the monomial.

- A **polynomial** is an algebraic expression which is the sum of finitely many monomials. If the polynomial consists of only one term, then it is a monomial, if it consist of $2, 3, 4$ terms, then we call it a **binomial, trinomial, quadrinomial**, respectively.

- The **degree of a polynomial** is the maximum of the degree of all monomials of the polynomial.

- A non-constant polynomial is called **univariate** if it contains a single variable, and **multivariate** otherwise.

- The quotient of two algebraic expressions (with non-zero denominator) is called a fractional expression. The quotient of two polynomials is called a **rational expression**.

- The **main term** of a univariate polynomial is its term with largest exponent. The coefficient of the leading term of a univariate polynomial is called the **leading coefficient** of the polynomial.

2.2 Polynomials

2.2.1 Basic operations on polynomials

Adding polynomials: To add two polynomials we build the result by including all the monomials of both polynomials and we simplify the result by collecting like terms.

Subtracting polynomials: Subtraction of polynomials is performed in the same way as addition, except that first we change the sign of all the monomials of the subtrahend polynomial.

Multiplying polynomials: To multiply two polynomials we multiply all terms of the firs multiplicand polynomial by all terms of the second multiplicand polynomial (one by one) and we simplify the result by collecting like terms.

Example. Let $P(x,y) := x^2 - 3xy + 2y^3$ and $Q(x,y) := 2x^2 - 3xy^2 + 3y^3$. Then we have

$$\begin{aligned}(P+Q)(x,y) =&(x^2 - 3xy + 2y^3) + (2x^2 - 3xy^2 + 3y^3) = \\ &= x^2 - 3xy + 2y^3 + 2x^2 - 3xy^2 + 3y^3 = 3x^2 - 3xy + 5y^3 - 3xy^2, \\ (P-Q)(x,y) =&(x^2 - 3xy + 2y^3) - (2x^2 - 3xy^2 + 3y^3) = \\ &= x^2 - 3xy + 2y^3 - 2x^2 + 3xy^2 - 3y^3 = -x^2 - 3xy - y^3 + 3xy^2, \\ (P\cdot Q)(x,y) =&(x^2 - 3xy + 2y^3)\cdot(2x^2 - 3xy^2 + 3y^3) = \\ &= \underbrace{2x^4 - 6x^3y + 4x^2y^3}_{(x^2-3xy+2y^3)\cdot 2x^2} \underbrace{-3x^3y^2 + 9x^2y^3 - 6xy^5}_{(x^2-3xy+2y^3)\cdot(-3xy^2)} + \underbrace{3x^2y^3 - 9xy^4 + 6y^6}_{(x^2-3xy+2y^3)\cdot 3y^3} = \\ &= 2x^4 - 3x^3y^2 - 6x^3y + 16x^2y^3 - 6xy^5 - 9xy^4 + 6y^6\end{aligned}$$

Exercise 2.1. Let $P(x,y), Q(x,y)$ and $H(x,y)$ be polynomials defined by

$$P(x,y) := x^2 - 3xy + y^2,$$
$$Q(x,y) := 2x^3 - x^2y + 3xy^2 + 5y^3,$$
$$H(x,y) := x^2 + 5xy + 3x - 2xy^3 + y.$$

Compute the following expressions:

a) $P(x,y) + H(x,y)$

b) $xP(x,y) + Q(x,y)$

c) $P(x,y) \cdot Q(x,y)$

d) $P(x,y) - 2H(x,y)$

e) $P(x,y) \cdot H(x,y)$

f) $(P(x,y) - H(x,y))P(x,y)$

Exercise 2.2. Let $P(x), Q(x)$ and $H(x)$ be univariate polynomials defined by

$$P(x) := x^2 - 3x + 2,$$
$$Q(x,y) := x^3 - 2x^2 + 3x + 5,$$
$$H(x,y) := x^2 + 2.$$

Compute the following expressions:

a) $P(x) + Q(x)$

b) $xP(x) + Q(x)$

c) $P(x) \cdot Q(x)$

d) $P(x) + 2H(x)$

e) $P(x) \cdot H(x)$

f) $(P(x) + Q(x)) \cdot H(x)$

g) $(x \cdot P(x) + 3 \cdot Q(x)) \cdot H(x)$

h) $P(x) + Q(x) + H(x)$

i) $P(x) \cdot (Q(x) + H(x))$

j) $P(x) \cdot Q(x) \cdot H(x)$

2.2.2 Division of polynomials by monomials

Division of a monomial by another monomial is done in the following steps

1). Divide the sign of the two monomials.

2). Divide the coefficients of the two monomials.

3). Divide the like variables by subtracting their exponents.

Example.

$$\frac{12x^4y^7z}{-3x^3y} = -\frac{12}{3} \cdot x^{4-3}y^{7-1}z^{1-0} = -4xy^6z$$

Exercise 2.3. Divide the following two monomials (with rational coefficients):

a) $12x^3y^4z$ by $3x^2yz$

b) $8x^2y^5z^3$ by $4x^2yz^2$

c) $2x^7y^2z^4$ by $4x^5z^3$

d) $3a^3b^4c^2$ by $5a^2c^2$

e) $6a^7b^5c^6$ by $3a^5bc^4$

f) $20a^5b^2c^9d^9$ by $25a^3b^2c^4d^7$

Division of a polynomial by a monomial is done in the following steps

1). Divide each term (monomial) of the dividend polynomial by the divisor monomial

2). Add the results to get the resulting polynomial

Example.

$$\frac{x^2y^3 - 3x^5y^2}{x^2y} = \frac{x^2y^3}{x^2y} + \frac{-3x^5y^2}{x^2y} = y^2 - 3x^3y.$$

Remark. Although the formal division of two monomials can be executed always, the result of this division is a monomial only when the dividend monomial is divisible by the divisor monomial. Otherwise the result is an algebraic expression, but not a monomial. The same is true for division of polynomials by monomials, and even for division of polynomials by polynomials.

2.2.3 Euclidean division of polynomials in one variable

In this section we present the division algorithm for univariate polynomials with complex, real or rational coefficients. This procedure is a straightforward generalization of the long division of integers.

The steps of the polynomial division:

1). Write the dividend and the divisor polynomials in the following scheme

dividend polynomial	divisor polynomial
	quotient polynomial

2). Divide the main term of the dividend by the main term of the divisor polynomial, and write the resulting monomial to the place of the quotient

3). Multiply the above resulting monomial by the divisor, change the sign of every monomial of the result, and write the resulting polynomial below the dividend.

4). Draw a horizontal line, add the dividend to the above resulting polynomial and write the result of the addition below the line

5). Let the polynomial below the last horizontal line take the role of the dividend and repeat steps 2)-4) until the polynomial below the last horizontal line is zero or has degree strictly smaller than the degree of the divisor.

Example. Divide the polynomial $f(x) = x^4-3x^3+5x^2+x-5$ by $g(x) = x^2-2x+2$.

$$
\begin{array}{rrrrr|l}
x^4 & -3x^3 & +5x^2 & +x & -5 & x^2-2x+2 \\ \cline{6-6}
-x^4 & +2x^3 & -2x^2 & & & x^2-x+1 \\ \cline{1-5}
 & -x^3 & +3x^2 & +x & -5 & \\
 & x^3 & -2x^2 & +2x & & \\ \cline{1-5}
 & & x^2 & +3x & -5 & \\
 & & -x^2 & +2x & -2 & \\ \cline{1-5}
 & & & 5x & -7 &
\end{array}
$$

Remark. If in the dividend polynomial there are missing terms of lower degrees, than it is wise to include them with coefficient 0 in the scheme, so that for their like terms there is place below them.

Example. Divide the polynomial $f(x) = x^4 + 5x^2 + x - 5$ by $g(x) = x^2 - 2x + 2$.

$$
\begin{array}{rrrrr|l}
x^4 & +0x^3 & +5x^2 & +x & -5 & x^2-2x+2 \\ \cline{6-6}
-x^4 & +2x^3 & -2x^2 & & & x^2+2x+7 \\ \cline{1-5}
 & 2x^3 & +3x^2 & +x & -5 & \\
 & -2x^3 & +4x^2 & -4x & & \\ \cline{1-5}
 & & 7x^2 & -3x & -5 & \\
 & & -7x^2 & +14x & -14 & \\ \cline{1-5}
 & & & 11x & -19 &
\end{array}
$$

Exercise 2.4. Divide the polynomial $f(x)$ by the polynomial $g(x)$ using the procedure of Euclidean division:

a) $f(x) := x^3 + 2x^2 - 4x + 2, \quad g(x) := x^2 - x + 1$

b) $f(x) := x^5 - 3x^4 + 4x^3 + 2x^2 - 4x + 2, \quad g(x) := x^2 - x + 1$

c) $f(x) := x^5 - 3x^4 + 2x^2 - 4x + 2, \quad g(x) := x^2 - 3x + 2$

d) $f(x) := x^6 - 3x^4 + 2x^2 - 4x + 2, \quad g := x^3 - 2x + 1$

e) $f(x) := x^5 - 3x^4 + 4x^3 - 5x^2 + x + 2 \quad g(x) := x^2 - 3x + 2$

f) $f(x) := x^6 - 64, \quad g(x) := x^2 - 2x + 4$

g) $f(x) := x^5 - 2x^4 - 3x + 2, \quad g(x) := x^2 - 3x + 4$

h) $f(x) := x^5 + 2x^4 - 5x^3 + 2, \quad g := x^2 - 5x + 2$

i $f(x) := x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 2, \quad g(x) := x^2 - 4x + 3$

j) $f(x) := x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 2, \quad g(x) := x^4 - 3x^3 + x^2 - 4x + 3$

k) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 1$

l) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 1$

m) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 2$

n) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 2$

o) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 3$

p) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x^2 - 1$

q) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 1$

r) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 1$

s) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 2$

t) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 2$

u) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 3$

v) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 3$

w) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - x - 1$

x) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - 1$

y) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + x - 1$

z) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + 2x - 1$

2.2.4 Horner's scheme

In this section we consider that case of the polynomial division, when the divisor takes the form $x - c$. Let us consider a concrete example:

$$
\begin{array}{rrrrr|l}
1x^4 & +2x^3 & +5x^2 & +x & -5 & x-2 \\ \cline{6-6}
-x^4 & +2x^3 & & & & 1x^3+4x^2+13x+27 \\ \cline{1-5}
 & 4x^3 & +5x^2 & +x & -5 & \\
 & -4x^3 & +8x^2 & & & \\ \cline{1-5}
 & & 13x^2 & +x & -5 & \\
 & & -13x^2 & +26x & & \\ \cline{1-5}
 & & & 27x & -5 & \\
 & & & -27x & +54 & \\ \cline{1-5}
 & & & & 49 &
\end{array}
$$

It is clear that in fact we only need to compute the numbers typeset by red, since they are exactly the coefficients of the quotient, and of course the remainder typeset in blue. The Horner's scheme below gives a much simpler procedure to compute these numbers:

Theorem 2.3. (Horner's scheme) *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *with* $a_n, \ldots, a_0 \in \mathbb{R}$, *and* $g(x) = x - c$ *with* $c \in \mathbb{R}$ *be two polynomials. Build the following table:*

	a_n	a_{n-1}	$\ldots\ldots$	a_{i+1}	$\ldots\ldots$	a_1	a_0
c	b_{n-1}	b_{n-2}	$\ldots\ldots$	b_i	$\ldots\ldots$	b_0	r

where

$$
\begin{aligned}
& b_{n-1} := a_n \\
& b_i := b \cdot b_{i+1} + a_{i+1} \qquad \text{for } i = n-2, n-3, \ldots, 0 \qquad (2.1) \\
& r := c \cdot b_0 + a_0.
\end{aligned}
$$

Then the quotient of the polynomial division of f *by* g *is the polynomial*

$$q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0,$$

and the remainder is the constant polynomial r. Further, we also have

$$f(c) = r. \tag{2.2}$$

Remark. In other words the above theorem states that in Horner's scheme the numbers in the first line are the coefficients of the polynomial f, and the numbers computed in the second line of the scheme are the following:

- the first number is the zero of the polynomial $g(x) = x - c$
- the next numbers (except for the last one) are the coefficients of the quotient polynomial of the polynomial division of f by g
- the last number is the remainder of the above division, but it is also the value of the polynomial $f(x)$ at $x = c$.

Example. Now we divide the polynomial $f(x) = x^4 + 2x^3 + 5x^2 + x - 5$ by the polynomial $x - 2$ using Horner's scheme. The first place in the first line is empty, then we list the coefficients of f. Then we solve the equation

$$x - 2 = 0$$

to get $x = 2$, thus the first element in the second line of the Horner's scheme will be 2. Then we compute the consecutive elements of the second line using (2.1) to get

	1	2	5	1	−5
2	1	4	13	27	49

This means that $f(x) = (x - 2)(x^3 + 4x^2 + 13x + 27) + 49$.

Remark. If in the dividend polynomial there are missing terms of lower degrees, than it is compulsory to include the coefficients of these terms (i.e. 0-s) in the upper row of the Horner's scheme.

Example. Divide the polynomial $f(x) = x^4 - 5x^2 + x - 5$ by the polynomial $x + 2$ using Horner's scheme. The first place in the first row is empty, then we list the coefficients of f, including the coefficient 0 of x^3. Then we solve the equation

$$x + 2 = 0$$

to get $x = -2$, thus the first element in the second row of the Horner's scheme will be 2. Then we compute the consecutive elements of the second line using (2.1) to get

	1	0	-5	1	-5
-2	1	-2	-1	3	-11

This means that $f(x) = (x + 2)(x^3 - 2x^2 - x + 3) + (-11)$.

Exercise 2.5. Divide the polynomial $f(x)$ by the monic linear polynomial $g(x)$

using Horner's scheme:

a) $f(x) = x^5 - 2x^4 + x^3 - 3x^2 + 2x - 5, \qquad g(x) = x + 1$

b) $f(x) = x^5 - 5x^4 + 3x^3 - 2x^2 + 2x - 3, \qquad g(x) = x - 2$

c) $f(x) = x^5 - 5x^4 + 3x^3 - 2x^2 + 2x - 3, \qquad g(x) = x + 2$

d) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x - 1$

e) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x + 1$

f) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x - 2$

g) $f(x) = x^7 - 5x^6 + 2x^5 - 4x^4 - x^3 + 3x - 2, \qquad g(x) = x + 2$

h) $f(x) = x^6 - 2, \qquad g(x) = x - 1$

i) $f(x) = x^6 - 2, \qquad g(x) = x - 2$

j) $f(x) = x^6 - 2, \qquad g(x) = x + 2$

k) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x - 1$

l) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 1$

m) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x - 2$

n) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 2$

o) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \qquad g(x) := x + 3$

p) $f(x) := x^7 - x^5 + x^3 - x, \qquad g(x) := x + 1$

q) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 1$

r) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 1$

s) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 2$

t) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 2$

u) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x - 3$

v) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \qquad g(x) := x + 3,$

w) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x + 1$

x) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x - 1$

y) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 4, \qquad g(x) = x + 2$

z) $f(x) = x^7 + 3x^6 + 2x^5 - 3x^4 + x^3 - 5x^2 - 3x + 6, \qquad g(x) = x + 1$

Exercise 2.6. Decide using Horner's scheme if the below polynomial $f(x)$ is divisible by the polynomial $g(x)$ or not, and if the answer is yes, then compute the quotient $\frac{f(x)}{g(x)}$, and if the answer is no, then compute the quotient and the remainder of the Euclidean division of $f(x)$ by $g(x)$:

a) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x - 2$

b) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x + 2$

c) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x - 1$

d) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x + 1$

e) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x + 3$

f) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x - 3$

g) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x + 4$

h) $f(x) = x^6 + 4x^5 - 11x^4 - 31x^3 - 4x^2 + 11x + 30, \qquad g(x) = x + 5$

i) $f(x) = x^7 + 3x^6 - 7x^5 - 28x^4 - 21x^3 + 7x^2 + 27x + 18, \qquad g(x) = x^2 - 1$

j) $f(x) = x^7 + 3x^6 - 7x^5 - 28x^4 - 21x^3 + 7x^2 + 27x + 18, \qquad g(x) = x^2 - 4$

k) $f(x) = x^7 + 3x^6 - 7x^5 - 28x^4 - 21x^3 + 7x^2 + 27x + 18, \qquad g(x) = x^2 - 9$

l) $f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x - 1$

m) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x + 1$

n) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x^2 - 1$

o) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x - 2$

p) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x + 2$

q) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x^2 - 4$

r) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x - 3$

s) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x + 3$

t) $f(x) = f(x) = x^7 + 6x^6 + 15x^5 + 21x^4 + 6x^3 - 15x^2 - 22x - 12, \qquad g(x) = x^2 - 9$

u) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \qquad g(x) = x + 1$

v) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \qquad g(x) = x - 1$

w) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \qquad g(x) = x^2 - 1$

x) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \qquad g(x) = x^2 - 4$

y) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \qquad g(x) = x^2 - 9$

z) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad g(x) = x^2 + 5x + 6$

Exercise 2.7. Compute the value of the following polynomial $f(x)$ at the given value of the indeterminate x:

a) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 1$

b) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 2$

c) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 3$

d) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -4$

e) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -3$

f) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -2$

g) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -1$

h) $f(x) = x^8 + x^7 - 11x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 0$

i) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 1$

j) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -1$

k) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 2$

l) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -2$

m) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 3$

n) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -3$

o) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 4$

2.3 Factorization of polynomials

A polynomial is in fact a sum of terms. Factorization of a polynomial is the process of finding polynomials whose product is the original polynomial.

We can consider multiplication of polynomials as the process of changing products into sums. In this sense factorization of polynomials is the inverse process, i.e. changing sums into products.

Polynomials that cannot be factorized are called **irreducible** polynomials.

The example of the polynomial $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$ shows that there are polynomials which can be factorized over the real numbers, meanwhile being irreducible over the rational numbers. So when we try to factorize a polynomial we have to specify what kind of coefficients are allowed for the factors.

Further, if we wish to factorize a polynomial over the rational or real numbers, then for every polynomial we clearly have infinitely many decompositions with one of the factors being a number (e.g. like $x + 1 = \frac{1}{2}(2x + 2)$). Thus such a decompositions is not considered a factorization.

2.3.1 Factorization by factoring out the greatest common monomial

If every monomial of the polynomial has a common factor, then we can factor out (or "pull out") this factor by using the distributive property of real numbers.

Example.

$$\underbrace{x^3y^2}_{xy^2\cdot(x^2)} + \underbrace{3xy^5}_{xy^2\cdot(-3y^3)} + \underbrace{x^5y^3}_{xy^2\cdot(x^4y)} = xy^2(x^2 - 3y^3 + x^4y)$$

2.3.2 Special products – Special factorization formulas

The following special product formulas in one hand give a help to shorten the computations with algebraic expressions, on the other hand (using them the other way around) they help at factorization of polynomials.

Powers of sums and differences

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i}b^i$$

Difference of perfect powers

$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$$

Sums of perfect powers

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^{2k+1} + b^{2k+1} = (a+b)(a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - \ldots - ab^{2k-1} + b^{2k})$$

Remark. We have not given formulas for the sum of perfect powers with even exponents. The reason for this is that there are no such formulas over the real numbers.

Example. Below we give some examples, where we use the above formulas to factorize polynomials.

- $x^4 - 4x^2y + 4y^2 = (x^2)^2 - 2 \cdot x^2 \cdot 2y + (2y)^2 = (x^2 - 2y)^2$

- $x^4 - 4y^2 = (x^2)^2 - (2y)^2 = (x^2 + 2y)(x^2 - 2y)$
- $x^4 - 6x^2y + 8y^2 = (x^2)^2 - 2x^2 \cdot 3y + (3y)^2 - y^2 = (x^2 - 3y)^2 - y^2 =$
  $(x^2 - 3y + y)(x^2 - 3y - y) = (x^2 - 2y)(x^2 - 4y)$

2.3.3 Factorization by grouping terms

If the polynomial to be factored has at least four terms (or it has three, but we can split one term in two), and neither factoring out a monomial, nor using special factorization formulas leads to a factorization, we may try to group the terms and factorize the groups so that, in the factorized form of all these groups we will find a common polynomial which can be factored out, this way leading to a factorized form of the original polynomial.

Example. Factorize the following polynomials over the integers:

1).

$$\begin{aligned} 3ab - 2ac + 6bd - 4cd =& (3ab - 2ac) + (6bd - 4cd) \\ &= a(3b - 2c) + 2d(3b - 2c) = (3b - 2c)(a + 2d) \end{aligned}$$

2).

$$\begin{aligned} x^3 + x^2y + xy^2 + y^3 =& (x^3 + x^2y) + (xy^2 + y^3) \\ &= x^2(x + y) + y^2(x + y) = (x + y)(x^2 + y^2) \end{aligned}$$

3). In this example, before the grouping we have to split one of the terms:

$$\begin{aligned} x^2 + 6x + 8 = x^2 + 4x + 2x + 8 =& (x^2 + 4x) + (2x + 8) \\ &= x(x + 4) + 2(x + 4) = (x + 4)(x + 2) \end{aligned}$$

2.3.4 General strategy of factorization of a polynomial

To factorize a polynomial one has to perform the following steps in the listed order.

1). Factor out the greatest common monomial factor.

2). Check if any of the special factorization formulas can be applied.

3). Try factoring by grouping terms.

4). If one succeeds in factoring the polynomial as the product of two or more polynomials, then it is necessary to restart the process for every factor.

5). The factorization is finished, when every factor turns out to be irreducible.

As the above strategy suggests, when factorizing polynomials in many cases, we have to combine the methods presented in Sections 2.3.1, 2.3.2 and 2.3.3.

Example. Factorize the following polynomials using integer coefficients:

1). In this example we first pull out the common factors of all monomials, then we group the terms, factorize the groups, then we pull out the common factor, finally we use the formula for difference of squares to factorize one of the resulting factors.

$$\begin{aligned} x^5y+x^4y^2 - x^3y^3 - x^2y^4 &= x^2y(x^3 + x^2y - xy^2 - y^3) \\ &= x^2y\left((x^3 + x^2y) - (xy^2 + y^3)\right) \\ &= x^2y\left(x^2(x + y) - y^2(x + y)\right) \\ &= x^2y(x + y)(x^2 - y^2) \\ &= x^2y(x + y)(x + y)(x - y) = x^2y(x + y)^2(x - y) \end{aligned}$$

2). In the present example we first group the terms, we recognize that the second group is a square of a binomial, then we use the formula for the difference of two squares.

$$\begin{aligned} x^4 - x^2 + 6x - 9 &= x^4 - (x^2 - 6x + 9) = (x^2)^2 - (x-3)^2 \\ &= \left(x^2 + (x-3)\right)\left(x^2 - (x-3)\right) = (x^2 + x - 3)(x^2 - x + 3) \end{aligned}$$

Exercise 2.8. Factorize the following polynomials into factors with rational coefficients:

a) $5x^2y + 15y - 5$	b) $8a - 8b + 16$	c) $8x^3 - 12x^2y^2 + 4x^2z$
d) $a^3b + a^2b^3 + a^2b$	e) $a(x+y) - ab(x+y)$	f)$2m^3 - 4m^5n + 2m^2$
g) $x^3y^2 - 5x^2y + x^4y^2$	h) $4a^2(b^2-2) - 2ab(b^2-2)$	i) $a(x-1) + b(1-x) - 7x + 7$
j) $x^2 - 25$	k) $x^2 - 9y^2$	l) $a^4 - 16$
m) $4a^2 - 12ab + 9b^2$	n) $a^3 - 6a^2b + 12ab^2 - 8b^3$	o) $125a^3 + 8b^3$
p) $81a^4 - 16b^2$	q) $(5x-3y)^2 - 81y^2$	r) $100(7x-3y)^2 - 9(4x+5y)^2$
s) $ac + ad + bc + bd$	t) $ac - ad + bc - bd$	u) $a^3 + 2a^2 + 2a + 4$
v) $a^3 + a^2b - ab^2 - b^3$	w) $x^3 - x^2z + 2xz^2 - 2z^3$	x) $a^4 + a^3b - ab^3 - b^4$
y) $x^2 - 4x + 3$	z) $x^2 - x - 6$	ω) $x^4 - 5x^2 + 4$

Exercise 2.9. Factorize the following polynomials into factors with rational coef-

ficients:

a) $3x + 18x^3y^3 + 27x^5y^6$

b) $x^3y^2 - 100x - x^2y^3 + 100y + x^2y^2z - 100z$

c) $9a^4 + 41a^2 - 20$

d) $(a^2b^2 + 1)^2 - (a^2 + b^2)^2$

e) $(x + 2y)^3 + (3x - y)^3$

f) $x^8 + x^4 + 1$

g) $(x + y)^4 + x^4 + y^4$

h) $x^4 - 2(a^2 + b^2)x^2 + a^4 + b^4 - 2a^2b^2$

i) $x^2 + 3x - x^4 - 3x$

j) $x^5 - 5x^4 + 4x^3 - x^2 + 5x - 4$

k) $x^2 + 2xy + y^2 - xz - yz$

l) $(abc + abd + acd + bcd)^2 - abcd(a + b + c + d)^2$

m) $3x^4y^4 - x^8 - y^8$

n) $(ac + pbd)^2 + p(ad - bc)^2$

o) $ac^2 - ab^2 + b^2c - c^3$

p) $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2$

q) $a^2b^4c^2 - a^2b^2c^4 + a^4b^2c^2 - a^4b^4$

r) $a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2 - 4abcd$

s) $x^5 + 2x^4 + 3x^2 + 2x + 1$

t) $9x^6 + 18x^5 + 26x^4 + 16x^3 + 6x^2 - 2x - 1$

u) $(x + y)^3 + 3(x + y)(x^2 - y^2) + 3(x - y)(x^2 - y^2) + (x - y)^3 - 27y^3$

v) $(cx + by)(ax + cy)(bx + ay) - (bx + cy)(cx + ay)(ax + by)$

w) $(x^2 + x + 1)(x^2 + x + 2) - 12$

x) $abc(a + b + c) - ab - ac - bc - a^2b^2c^2 + 1$

y) $(x^2 + x + 1)(x^3 + x^2 + 1) - 1$

z) $(x - a)^3(b - c) + (x - b)^3(c - a) + (x - c)^3(a - b) + 3x(b - c)(c - a)(a - b)$

2.3.5 Divisibility of polynomials

Definition 2.4. Let $\mathbb{T}$ be any of the sets $\mathbb{Q}, \mathbb{R}$. Let $P(x), Q(x) \in \mathbb{T}[x]$ be two polynomials. We say that Q **divides** P if there exists a polynomial $R(x) \in \mathbb{T}[x]$ such that $P(x) = Q(x)R(x)$. Further, if Q divides P we also say that P **is divisible by** Q, or Q **is a factor of** P.

Notation. For Q **divides** P we use the notation $Q \mid P$ or $Q(x) \mid P(x)$.

Remark. The fact that Q is a divisor of P in fact means that if we perform the Euclidean division of P by Q then the remainder is 0.

Theorem 2.5. (Properties of the divisibility of polynomials with rational coefficients) *Let $P, Q, R, S, T \in \mathbb{Q}[x]$ be polynomials with rational coefficients. Then we have the following:*

1). $P \mid P$

2). if $P \mid Q$ and $Q \mid P$ then we have $P = aQ$ with some $a \in \mathbb{Q} \setminus \{0\}$

3). if $P \mid Q$ and $Q \mid R$ then $P \mid R$

4). if $P \mid Q$ then $P \mid QR$

5). if $P \mid Q$ and $P \mid R$ then $P \mid (P \pm Q)$

6). if $P \mid Q$ and $P \mid R$ then $P \mid (S \cdot P \pm T \cdot Q)$

2.3.6 Roots of polynomials

Theorem 2.6. (The remainder theorem) *Let $P(x) \in \mathbb{R}[x]$ be a polynomial and $r \in \mathbb{R}$ a real number. Then $P(r)$ is just the remainder of the Euclidean division of $P(x)$ by $x - r$.*

Definition 2.7. If $P(x) \in \mathbb{R}[x]$ is a polynomial with real coefficients then we say that the number $r \in \mathbb{R}$ is a root of P if $P(r) = 0$.

Theorem 2.8. (The factor theorem) *Let $P(x) \in \mathbb{R}[x]$ be a polynomial with real coefficients and $r \in \mathbb{R}$ a real number. Then r is a root of P if and only if $x - r$ is a factor of $P(x)$, i.e. $P(x) = (x - r)Q(x)$ with some polynomial $Q(x) \in \mathbb{R}[x]$*

2.4 Rational algebraic expressions

Definition 2.9. A **rational expression** is a quotient of polynomials.

Example.

$$\frac{x^3+x+1}{x-7}, \quad \frac{1}{x} \quad x^2+3$$

2.4.1 Simplification and amplification of rational algebraic expressions

> If the numerator and the denominator of a rational expression has a common factor then we may cancel this common factor from both the numerator and the denominator, and the resulting expression will be an equivalent expression to the original one.

Example.

$$\frac{x^2-1}{(x+2)(x-1)} = \frac{(x+1)(x-1)}{(x+2)(x-1)} = \frac{x+1}{x+2}$$

Remark. It is very important that we can only simplify by a factor which is a factor of the whole numerator and the whole denominator, but we cannot simplify by an expression which is a factor of only a term of the numerator or (denominator), but not of the whole numerator (or denominator). For example, in the fraction below we cannot simplify by $x+1$:

$$\frac{2x^2+(x+3)(x+1)}{(x+1)(x^2+3)}.$$

Exercise 2.10. Simplify the following expressions

a) $\dfrac{30x^2y}{6xy^2}$ b) $\dfrac{5x^2y^7z^3}{10x^4yw^2}$

c) $\dfrac{-12x^4}{24x^6}$ d) $\dfrac{3(-x)^5}{12(-x)^6}$

e) $\dfrac{x-3}{2(x-3)^2}$ f) $\dfrac{5(x-1)^2}{x^2-1}$

g) $\dfrac{9x^2+3xy}{3xy+9y^2}$ h) $\dfrac{x^2-8x}{x^3-8x^2}$

i) $\dfrac{5x-20}{x^2-16}$ j) $\dfrac{7x^4-7y^4}{9x^2y^2+9y^4}$

We may multiply both the numerator and the denominator of an algebraic fraction by a non-zero expression, and we get a fraction which is equivalent to the original fraction.

Example. Here we amplify the fraction $\frac{x+1}{x-1}$ by x^2+x+1:

$$\frac{x+1}{x-1} = \frac{(x+1)(x^2+x+1)}{(x-1)(x^2+x+1)} = \frac{(x+1)(x^2+x+1)}{x^3-1}.$$

Exercise 2.11. Amplify the following expressions so that they have the same denominator

a) $\dfrac{3}{5a^2b^7}$ and $\dfrac{1}{a^3b}$

b) $\dfrac{1}{a^3(x+1)}$ and $\dfrac{1}{a^2(x+1)^2}$

c) $\dfrac{x+3}{2x-1}$ and $\dfrac{x-1}{3x+2}$

d) $\dfrac{1}{x+1}$, $\dfrac{1}{a^3}$ and $\dfrac{1}{3a}$

2.4.2 Multiplication and division of rational expressions

- If we have to **multiply** two rational algebraic expressions, we have to multiply the numerator by the numerator and the denominator by the denominator, however, if possible, we first simplify.
- If we have to **divide** two rational algebraic expressions, we multiply the dividend expression by the reciprocal of the divisor expression.

Example. Here we present a simple example of multiplication of two rational expressions

$$\frac{(x+1)(x^2+x+1)}{x+3} \cdot \frac{x+5}{(x+4)(x+1)} = \frac{(x^2+x+1)(x+5)}{(x+3)(x+4)}$$

Example. Here we present a simple example of division of two rational expressions

$$\frac{(x-1)(x^2+1)}{x-3} : \frac{(x-1)(x+5)}{x+4} = \frac{(x-1)(x^2+1)}{x-3} \cdot \frac{x+4}{(x-1)(x+5)} = \frac{(x^2+1)(x+4)}{(x-3)(x+5)}$$

2.4.3 Addition and subtraction of rational algebraic expressions

1. If we have to **add (subtract)** two or more rational algebraic expressions, **whose denominators are the same**, then the result is a fraction whose denominator is the same like the common denominator of the summands, and the numerator is the sum (difference) of the original numerators.
2. If we have to **add (subtract)** two or more rational algebraic expressions, **whose denominators are different** then we first amplify the fractions so that all denominators become the same expression, and we use the rule described in 1.
3. When choosing the common denominator, we have to try to find the most simple such expression, i.e the least common multiple of all the denominators.

The above rules can be summarized by the above formulas. If the denominators are the same, then

$$\frac{a}{d}+\frac{c}{d}=\frac{a+c}{d}, \qquad \frac{a}{d}-\frac{c}{d}=\frac{a-c}{d}.$$

In the case when the denominators are different, then

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd},$$
$$\frac{a}{be}+\frac{c}{de}=\frac{ad}{bde}+\frac{bc}{bde}=\frac{ad+bc}{bde}$$

Remark. The product of the denominators is always a theoretically possible choice for the common denominator, however we strongly discourage the student to choose this way, since it may make the solution much more complicated. It is always advised to search for the simplest possible common denominator (i.e. the least common multiple of the denominators of the summands), even if this requires the factorization of the denominators of the summands.

Example. In the following example we may just take the product of the denominators as the common denominator:

$$\begin{aligned}\frac{x}{x-1}+\frac{x+1}{x+2} &= \frac{x(x+2)}{(x-1)(x+2)}+\frac{(x+1)(x-1)}{(x+2)(x-1)} = \\ &= \frac{x(x+2)+(x+1)(x-1)}{(x-1)(x+2)} \\ &= \frac{x^2+2x+x^2-1}{(x-1)(x+2)} = \frac{2x^2+2x-1}{(x-1)(x+2)}.\end{aligned}$$

In the next example (as always) it is also possible to take the product of the denominators as the common denominator, however, this makes the computations much more complicated. **We strongly suggest not to choose the following solution** for this exercise, however we include it to show the difference between this, and the simplest solution:

$$\begin{aligned}\frac{x}{x^2-1}+\frac{x+3}{(x-1)(x+2)} &= \frac{x(x-1)(x+2)}{(x^2-1)(x-1)(x+2)}+\frac{(x+3)(x^2-1)}{(x^2-1)(x-1)(x+2)} \\ &= \frac{x^3+x^2-2x}{(x^2-1)(x-1)(x+2)}+\frac{x^3+3x^2-x-3}{(x-1)(x+2)(x^2-1)} \\ &= \frac{x^3+x^2-2x+x^3+3x^2-x-3}{(x^2-1)(x-1)(x+2)} = \frac{2x^3+4x^2-3x-3}{(x+1)(x-1)^2(x+2)} \\ &= \frac{(2x^3-2x^2)+(6x^2-6x)+(3x-3)}{(x+1)(x-1)^2(x+2)} \\ &= \frac{2x^2(x-1)+6x(x-1)+3(x-1)}{(x+1)(x-1)^2(x+2)} \\ &= \frac{(x-1)(2x^2+6x+3)}{(x+1)(x-1)^2(x+2)} = \frac{(2x^2+6x+3)}{(x+1)(x-1)(x+2)}\end{aligned}$$

Now we give the simplest solution to the above example. This shows that it is always important to try to find the least common multiple of the denominators and use that expression as the common denominator, even if this needs the factorization of the denominators of the summands.

$$\begin{aligned}\frac{x}{x^2-1}+\frac{x+3}{(x-1)(x+2)} &= \frac{x}{(x+1)(x-1)}+\frac{x+3}{(x-1)(x+2)}\\ &= \frac{x(x+2)}{(x+1)(x-1)(x+2)}+\frac{(x+3)(x+1)}{(x+1)(x-1)(x+2)}\\ &= \frac{x^2+2x}{(x+1)(x-1)(x+2)}+\frac{x^2+3x+x+3}{(x-1)(x+2)(x^2-1)}\\ &= \frac{x^2+2x+x^2+3x+x+3}{(x+1)(x-1)(x+2)}=\frac{(2x^2+6x+3)}{(x+1)(x-1)(x+2)}\end{aligned}$$

In exercises generally our main goal is to transform complicated algebraic expressions to an equivalent, but much simpler algebraic expression.

Example. Simplify the following algebraic expression:

$$\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right)\right]:\frac{a^3-b^3}{a^2b^2}\qquad a\neq 0, b\neq 0, a\neq -b.$$

Solution:

$$\begin{aligned}&\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right)\right]:\frac{a^3-b^3}{a^2b^2}\\ &=\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a-b}\left(\frac{b}{ab}-\frac{a}{ab}\right)\right]\cdot\frac{a^2b^2}{a^3-b^3}\\ &=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a-b}\cdot\frac{b-a}{ab}\right)\cdot\frac{a^2b^2}{a^3-b^3}\\ &=\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{ab}\right)\cdot\frac{a^2b^2}{a^3-b^3}\\ &=\left(\frac{b^2}{a^2b^2}+\frac{a^2}{a^2b^2}-\frac{2ab}{a^2b^2}\right)\cdot\frac{a^2b^2}{a^3-b^3}\\ &=\frac{b^2+a^2-2ab}{a^2b^2}\cdot\frac{a^2b^2}{a^3-b^3}\\ &=\frac{(a-b)^2}{a^2b^2}\cdot\frac{a^2b^2}{(a-b)(a^2+ab+b^2)}\\ &=\frac{a-b}{a^2+ab+b^2}\end{aligned}$$

Exercise 2.12. Simplify the following algebraic expressions

a) $\dfrac{2a+1}{a+1}+\dfrac{a-2}{a-1}-\dfrac{3a^2-1}{a^2-1}$

b) $\left(\dfrac{2}{1-x^2}+\dfrac{x+2}{x-1}\right):\dfrac{x+3}{x^2-1}$

c) $\dfrac{ax-6+3x-2a}{ax+6-3x-2a}$

d) $\dfrac{a^2x^2-x^2b^2+2a^2x-2b^2x+a^2-b^2}{(a-b)(x+1)}$

e) $\dfrac{xy^2+2x^2y^2+x^3y+xy^2(x+y)^2}{xy^2-x-y(1-y^2)}$

f) $\dfrac{x^3+x^2y-x-y}{(x^2+2xy+y^2)(x-1)}$

g) $\dfrac{x^2(x^2+a^2)^2}{\frac{x^{10}-a^{10}}{x^2-a^2}-\frac{x^{12}+a^{12}}{x^4+a^4}}$

h) $\dfrac{a^3x^2+b^3x^2+3abx^2(a+b)+(a+b)^3}{a^2x^2+2abx^2+b^2x^2+(a+b)^2}$

i) $\left[\left(\frac{2a}{b}-\frac{b}{2a}\right)^2+2\right]\frac{2ab}{16a^4+b^4}$

j) $\left(\frac{5a}{a+x}+\frac{5x}{a-x}+\frac{10ax}{a^2-x^2}\right):\left(\frac{a}{a+x}+\frac{x}{a-x}+\frac{2ax}{a^2-x^2}\right)$

k) $\left(\frac{b^2}{a^3-ab^2}+\frac{1}{a+b}\right):\left(\frac{a-b}{a^2+ab}-\frac{a}{b^2+ab}\right)$

l) $\left(x-\frac{4xy}{x+y}+y\right):\left(\frac{x}{x+y}-\frac{y}{y-x}-\frac{2xy}{x^2-y^2}\right)$

m) $\left(\frac{(a+b)^2+2b^2}{a^3-b^3}-\frac{1}{a-b}+\frac{a+b}{a^2+ab+b^2}\right)\cdot\left(\frac{1}{b}-\frac{1}{a}\right)$

n) $\frac{x^2-1}{xy}:\left[\left(\frac{x^2-xy}{x^2y+y^3}-\frac{2x^2}{y^3-xy^2+x^2y-x^3}\right)\cdot\left(1-\frac{y-1}{x-\frac{y}{x^2}}\right)\right]$

o) $\frac{x^3-\left(\frac{1}{1+\frac{1}{x}}+\frac{1}{\frac{1}{x}+\frac{1}{x^2}}\right):\left(\frac{1}{\frac{1}{x}-1}-\frac{1}{\frac{1}{x^2}-\frac{1}{x}}\right)}{x^3-1}$

p) $\frac{\left(\frac{x+y^2}{a+b}-\frac{x^4-y^4}{a^3+b^3}:\frac{x^2-y^2}{a^2-ab+b^2}\right)\cdot(2a+2b-ax-bx)}{x^2-3x+3}$

q) $\frac{\frac{a^2+ab+b^2}{a^3-b^3}-\frac{a^2+2ab+b^2}{a^3+b^3}:\frac{a^2-b^2}{a^2-ab+b^2}}{(7a^3+ab+b^5)(2a-3b)-b^3+a^3}$

2.5 Algebraic expressions containing roots

When working with expressions containing roots the main difference to the case of rational expressions is that it is harder to find a suitable but simple common denominator. Thus in many cases it is useful to **rationalize the denominator** of such fractions (i.e. to get rid of the roots appearing in the denominator using equivalent transformations of the expression). This is done generally by using formulas for special products.

Here we present the most frequently used methods for rationalizing the denominator of a fraction:

1). If the denominator is a one-term expression containing only one nth root, then we amplify the fraction by the $(n-1)$th power of that root. Specially, if the one-term denominator contains only one square-root, then we amplify the fraction by that square-root. Indeed,

$$\frac{1}{\sqrt[n]{a}} = \frac{\left(\sqrt[n]{a}\right)^{n-1}}{\sqrt[n]{a}\cdot\left(\sqrt[n]{a}\right)^{n-1}} = \frac{\left(\sqrt[n]{a}\right)^{n-1}}{a},$$

and in the special case of a square-root

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

2). If the denominator is the sum or difference of two square-roots (one of those being possibly the square-root of a perfect square), then we use the formula for difference of two squares, namely

$$x^2 - y^2 = (x+y)(x-y).$$

Indeed, if we wish to rationalize the denominator of $\frac{1}{\sqrt{a}-\sqrt{b}}$ $(a, b > 0)$ then we amplify the fraction by $\sqrt{a}+\sqrt{b}$ as follows:

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

Similarly, we have

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

3). If the denominator is the sum or the difference of two cube-roots (one of those being possibly the cube-root of a perfect cube), then we use one of the formulas

$$x^3 - y^3 = (x-y)(x^2+xy+y^2) \quad \text{or} \quad x^3 + y^3 = (x+y)(x^2-xy+y^2).$$

Indeed, if we wish to rationalize the denominator of $\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}}$ $(a, b > 0)$ then we amplify the fraction by $(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2$ as follows:

$$\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}} = \frac{(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left((\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2\right)} = \frac{(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{a-b}$$

Similarly, we have

$$\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}} = \frac{(\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left((\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2\right)} = \frac{(\sqrt[3]{a})^2 - \sqrt[3]{a}\sqrt[3]{b} + \left(\sqrt[3]{b}\right)^2}{a+b}$$

Example. Rationalize the denominator of the following fractions

1). $\frac{3}{2\sqrt{5}}$

$$\frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}} = \frac{3\sqrt{5}}{10}$$

2). $\frac{1}{2\sqrt[5]{3}}$

$$\frac{1}{2\sqrt[5]{3}} = \frac{\left(\sqrt[5]{3}\right)^4}{2\sqrt[5]{3}\cdot\left(\sqrt[5]{3}\right)^4} = \frac{\sqrt[5]{81}}{6}$$

3). $\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$

$$\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{6}(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{\sqrt{6}(\sqrt{2}-\sqrt{3})}{2-3} = \sqrt{6}(\sqrt{3}-\sqrt{2}) = 3\sqrt{2}-2\sqrt{3}$$

4). $\frac{1}{2-\sqrt{3}}$

$$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

5). $\frac{1}{\sqrt[3]{5}+\sqrt[3]{2}}$

$$\frac{1}{\sqrt[3]{5}+\sqrt[3]{2}} = \frac{\left(\sqrt[3]{5}\right)^2 - \sqrt[3]{5}\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2}{\left(\sqrt[3]{5}+\sqrt[3]{2}\right)\left(\left(\sqrt[3]{5}\right)^2 - \sqrt[3]{5}\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2\right)}$$

$$= \frac{\left(\sqrt[3]{5}\right)^2 - \sqrt[3]{5}\sqrt[3]{2} + \left(\sqrt[3]{2}\right)^2}{5+2} = \frac{\left(\sqrt[3]{25}\right) - \sqrt[3]{10} + \left(\sqrt[3]{4}\right)}{7}$$

Exercise 2.13. Rationalize the denominator of the following expressions:

a) $\frac{1}{\sqrt{7}}$ b) $\frac{1}{\sqrt[3]{5}}$ c) $\frac{1}{\sqrt[k]{a}}$

Exercise 2.14. Rationalize the denominator of the following expressions:

a) $\frac{1}{\sqrt{5}-\sqrt{2}}$ b) $\frac{1}{1-\sqrt{5}}$ c) $\frac{1}{\sqrt{7}+\sqrt{3}}$

d) $\frac{3}{\sqrt{5}+\sqrt{2}}$ e) $\frac{5}{\sqrt{6}+1}$ f) $\frac{3}{-\sqrt{11}+\sqrt{13}}$

g) $\frac{1}{\sqrt[3]{5}-\sqrt[3]{2}}$ h) $\frac{3}{\sqrt[3]{7}-\sqrt[3]{4}}$ i) $\frac{1}{\sqrt[3]{5}+\sqrt[3]{2}}$

j) $\frac{1}{\sqrt[4]{5}-\sqrt[4]{2}}$ k) $\frac{1}{\sqrt[5]{5}-\sqrt[5]{2}}$ l) $\frac{1}{\sqrt{5}-\sqrt[3]{2}}$

m) $\frac{1}{\sqrt[4]{5}+\sqrt[4]{2}}$ n) $\frac{1}{\sqrt{5}-\sqrt{3}+\sqrt{2}}$ o) $\frac{1}{\sqrt{5}-\sqrt{3}-\sqrt{2}}$

p) $\frac{1}{\sqrt{5-\sqrt{2}}+\sqrt{5+\sqrt{2}}}$ q) $\frac{7}{1-\sqrt[4]{2}+\sqrt{2}}$ r) $\frac{1-\sqrt{2}+\sqrt[3]{2}}{1+\sqrt{2}-\sqrt[3]{2}}$

s) $\frac{1}{\sqrt{a}+\sqrt{b}}$ t) $\frac{1}{\sqrt{a}-\sqrt{b}}$ u) $\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$

v) $\frac{1}{\sqrt{a}+\sqrt[3]{b}}$ w) $\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}}$ x) $\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}}$

y) $\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}$ z) $\frac{1}{\sqrt[n]{a}-\sqrt[n]{b}}$ α) $\frac{1}{\sqrt[12]{a}+\sqrt[12]{b}}$

β) $\frac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}$ γ) $\frac{1}{1-\sqrt[3]{2}+\sqrt[3]{4}}$ δ) $\frac{1}{\sqrt{2+\sqrt{2+\sqrt{2}}}}$

Example. Compute the exact value of the following expression

$$S := \sqrt{4+2\sqrt{3}} - \sqrt{4-2\sqrt{3}}.$$

Solution 1. Our first solution uses the idea that below both square roots there is a perfect square, which can be observed using the formulas

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)^2 = a^2 - 2ab + b^2,$$

and trying to find numbers a and b with $2ab = 2\sqrt{3}$ and $a^2 + b^2 = 4$. So we have

$$\begin{aligned} S &= \sqrt{4+2\sqrt{3}} - \sqrt{4-2\sqrt{3}} = \sqrt{3+2\sqrt{3}+1} - \sqrt{3-2\sqrt{3}+1} = \\ &= \sqrt{\sqrt{3}^2 + 2\cdot\sqrt{3}\cdot 1 + 1^2} - \sqrt{\sqrt{3}^2 + 2\cdot\sqrt{3}\cdot 1 + 1^2} = \sqrt{\left(\sqrt{3}+1\right)^2} - \sqrt{\left(\sqrt{3}-1\right)^2} \\ &= \left|\sqrt{3}+1\right| - \left|\sqrt{3}-1\right| = \left(\sqrt{3}+1\right) - \left(\sqrt{3}-1\right) = \sqrt{3}+1-\sqrt{3}+1 = 2. \end{aligned}$$

Solution 2. The second solution is based on the idea to compute the square of S. We have

$$\begin{aligned} S^2 &= \left(\sqrt{4+2\sqrt{3}} - \sqrt{4-2\sqrt{3}}\right)^2 \\ &= \left(\sqrt{4+2\sqrt{3}}\right)^2 - 2\sqrt{4+2\sqrt{3}}\cdot\sqrt{4-2\sqrt{3}} + \left(\sqrt{4-2\sqrt{3}}\right)^2 \\ &= 4+2\sqrt{3} - 2\sqrt{(4+2\sqrt{3})\cdot(4-2\sqrt{3})} + 4 - 2\sqrt{3} \\ &= 8 - 2\sqrt{4^2-(2\sqrt{3})^2} = 8 - 2\sqrt{16-12} = 8 - 2\sqrt{4} = 4. \end{aligned}$$

Further, since $\sqrt{4+2\sqrt{3}} > \sqrt{4-2\sqrt{3}}$ we have $S > 0$. So we have proved that

$$\left.\begin{array}{l} S^2 = 4 \\ S > 0 \end{array}\right\} \Longrightarrow S = 2.$$

Exercise 2.15. Compute the exact value of the following expressions:

a) $\sqrt{6+2\sqrt{5}}-\sqrt{6-2\sqrt{5}}$

b) $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$

c) $\sqrt{16+8\sqrt{3}}-\sqrt{16-8\sqrt{3}}$

d) $\sqrt{9+3\sqrt{5}}\cdot\sqrt{9+3\sqrt{5}}$

e) $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$

f) $\sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}}+\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}$

g) $(\sqrt{3}+3)\sqrt[3]{54-30\sqrt{3}}$

h) $\sqrt{17-4\sqrt{9+4\sqrt{5}}}$

i) $2\sqrt{3+\sqrt{5-\sqrt{13+\sqrt{48}}}}$

j) $\sqrt{13+30\sqrt{2+\sqrt{9+4\sqrt{2}}}}$

k) $\sqrt{26+6\sqrt{13-4\sqrt{8+2\sqrt{6-2\sqrt{5}}}}}-\sqrt{26-6\sqrt{13+4\sqrt{8-2\sqrt{6+2\sqrt{5}}}}}$

l) $\sqrt{2+\sqrt{3}}\sqrt{2+\sqrt{2+\sqrt{3}}}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$

m) $1+\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{2011}+\sqrt{2012}}$

n) $\dfrac{\sqrt{\sqrt[4]{8}+\sqrt{\sqrt{2}-1}}-\sqrt{\sqrt[4]{8}+\sqrt{\sqrt{2}-1}}}{\sqrt{\sqrt[4]{8}-\sqrt{\sqrt{2}+1}}}$

Exercise 2.16. Simplify the following expressions:

a) $\dfrac{3\sqrt{a}-2}{\sqrt{a}+1}-\dfrac{2\sqrt{a}-4}{\sqrt{a}-1}-\dfrac{a+1}{a-1}$

b) $\left(\dfrac{2a+3\sqrt{a}}{4a+12\sqrt{a}+9}-\dfrac{3\sqrt{a}+2}{2\sqrt{a}+3}+\dfrac{4\sqrt{a}-1}{2\sqrt{a}+3}\right)\dfrac{2\sqrt{a}+3}{2\sqrt{a}-3}$

c) $\dfrac{\sqrt{x}+\sqrt{y}-1}{x+\sqrt{xy}}+\dfrac{\sqrt{x}-\sqrt{y}}{2\sqrt{xy}}\left(\dfrac{\sqrt{y}}{x-\sqrt{xy}}+\dfrac{\sqrt{y}}{x+\sqrt{xy}}\right)$

d) $\left(\dfrac{a+\sqrt{ab}}{\sqrt{a^3}+a\sqrt{b}+b\sqrt{a}+\sqrt{b^3}}+\dfrac{\sqrt{b}}{a+b}\right):\left(\dfrac{1}{\sqrt{a}-\sqrt{b}}-\dfrac{2\sqrt{ab}}{a\sqrt{a}-a\sqrt{b}+b\sqrt{a}-b\sqrt{b}}\right)$

Chapter 3

Equations I

3.1 Introduction to equations

Definition 3.1. An **equation** is a statement that two algebraic expressions are equal.

Example. By the above definition the statement

$$\frac{1}{\sqrt{3}-\sqrt{2}} = \sqrt{3}+\sqrt{2}$$

is an equation, however, in algebra we are more interested in equations which contain variables (i.e. letters).

Example. Here are three examples of equations which look completely similar:

1). $3x + 4 - 3(x+3) + 5 = 0$,

2). $3x + 4 - 3(x+3) + 5 = 1$,

3). $3x + 4 - 2(x+3) + 2 = 0$.

However, if we simplify the left hand sides of the above equations we get

1). $0 = 0$,

2). $0 = 1$,

3). $x = 0$.

So we can see that we are in three completely different situations:

1). the equation is **true for all possible values** of x,

2). the equation is **false for all possible values** of x,

3). the equation is true for only a single value of x, namely 0, so the **truth of the equation depends on the value** of x.

The above example justifies to give separate names for such equations, as we shall do it in the definitions below.

Definition 3.2. An equation which is always false, independently of the value of the variables involved, is called a **contradiction**.

Definition 3.3. An equation which is always true, independently of the value of the variables involved, is called an **identity**.

Definition 3.4. An equation whose truth depends on the value of the variables involved is called a **conditional equation**. To **solve** a conditional equation means to find those values of the variables for which the equation is true. These values are called **solutions** (or **roots**) of the equation. To **check the solutions** of the equation means that we insert the values of the possible solutions one-by-one into the original equation and check if the resulting equation is true or not. Only those values are accepted as solutions of the original equation which give a true equality when checked.

Definition 3.5. Two equations that have the same solutions are called **equivalent equations**. Transformations that modify an equation to an equivalent equation are called **equivalent transformations** of the equation. We say that two equations are in **consequential relation** to each other, if all solutions of the first one are also solutions to the other one.

Equivalent Transformations of Equations

1). Equivalent transformation of the algebraic expression on the left hand side

2). Equivalent transformation of the algebraic expression on the right hand side

3). Adding the same number or algebraic expression to both sides

$$a = b \iff a + c = b + c$$

4). Subtracting the same number or algebraic expression from both sides

$$a = b \iff a - c = b - c$$

5). Multiplying both sides by the same non-zero number or non-zero algebraic expression

$$a = b \iff ac = bc \text{ with } c \neq 0$$

6). Dividing both sides by the same non-zero number or non-zero algebraic expression

$$a = b \iff \frac{a}{c} = \frac{b}{c} \text{ with } c \neq 0$$

Remark. The restriction $c \neq 0$ when multiplying an equation by c is important in order to get an equivalent equation. If we omit this restriction, then the resulting equation will posses all solutions of the original equation (i.e. the two equations will be in consequential relation), however it may also posses solutions which are not solutions to the original equation. Such solutions are called ??? solutions.

Convention. Whenever writing a set of equations below each other we make the convention that this means that they are in consequential relation, unless said otherwise. So it is **compulsory to check the solutions**.

In many countries the convention is different, i.e. the equations written below each other are equivalent to each other, and then it is not necessary to check the solutions. However, since in Hungary we use the above convention, thus it is necessary to check the solutions, unless we mention at the beginning of the solution that we are doing equivalent transformations.

For the rest of this chapter we restrict our investigation to equations in one variable. In the sequel (in this Chapter and Chapter 5) we present methods to solve several kinds of equations, like linear equations, quadratic equations, equations containing absolute values, irrational equations, etc.

3.2 Linear Equations

Definition 3.6. An equation is called a **linear equation** (or **an equation of degree one**) if it is equivalent to an equation of the form

$$ax + b = 0 \quad \text{where } a, b \text{ are constants, and } a \neq 0.$$

Method of solving linear equations

1). Multiply the equation by the least common multiple of all denominators appearing in the equation.

2). Simplify the algebraic expressions on both sides of the equation.

3). Add and subtract suitable expressions so that all terms containing the variables be on the left hand side, and all the terms free of the variables to the right hand side.

4). Add the like terms in both sides to get an equation of the shape $cx = d$

5). If the coefficient c of x in the left hand side is non-zero, then divide by it, to get the single solution of the equation, otherwise decide if the equation is a contradiction or an identity.

6). If there is a single solution check that solution, if there are infinitely many solutions put the necessary conditions and mention that we have used only equivalent transformations.

Example. • Solve the following linear equation over the real numbers

$$\frac{x-2}{3} + \frac{x}{4} = 4$$

Solution. We follow the general strategy described above:

$$\begin{aligned} &\frac{x-2}{3}+\frac{x}{4}=4 && /\cdot 12 \\ &4(x-2)+3x=48 \\ &4x-8+3x=48 \\ &7x-8=48 && /\ +8 \\ &7x=56 && /\ :7 \\ &x=8 \end{aligned}$$

Finally, we have to check the result:

$$\begin{aligned} &\frac{8-2}{3}+\frac{8}{4}=4 \\ &2+2=4 \ \text{ true} \end{aligned}$$

So the solutionset is $S=\{8\}$.

- Solve the following linear equation over the real numbers

$$\frac{x-2}{3}+\frac{x}{4}=4+\frac{7x}{12}$$

Solution. We follow the general strategy described above:

$$\begin{aligned} &\frac{x-2}{3}+\frac{x}{4}=4+\frac{7x}{12} && /\cdot 12 \\ &4(x-2)+3x=48+7x \\ &4x-8+3x=48+7x \\ &7x-8=48+7x && /\ +8-7x \\ &0=56 \quad \textbf{contradiction} \end{aligned}$$

So there is no solution to this equation, i.e. the solutionset is the emptyset: $S=\emptyset$.

- Solve the following linear equation over the real numbers

$$\frac{x-2}{3} + \frac{x}{4} = \frac{7x-8}{12}$$

Solution. We follow the general strategy described above:

$$\frac{x-2}{3} + \frac{x}{4} = \frac{7x-8}{12} \qquad / \cdot 12$$

$$4(x-2) + 3x = 7x - 8$$

$$4x - 8 + 3x = 7x - 8$$

$$7x - 8 = 7x - 8 \qquad / \ +8 - 7x$$

$$0 = 0 \quad \textbf{identity}$$

We have used equivalent transformations, and our equation reduces to an identity, which means that every real number for which the original equation is defined (i.e. the expressions in the equation have sense) is a solution to this equation. In our case every expression in the original equation has sense for every real number.

This means the solutionset is the whole set of real numbers: $S = \mathbb{R}$.

- Solve the following linear equation over the real numbers

$$\frac{x-2}{3x} + \frac{1}{4} = \frac{7x-8}{12x}$$

Solution. First we have to put conditions:

$$3x \neq 0 \quad \text{and} \quad 12x \neq 0$$

which leads to $x \in \mathbb{R} \setminus \{0\}$.

Now we follow the general strategy described above:

$$\frac{x-2}{3x} + \frac{1}{4} = \frac{7x-8}{12x} \qquad / \cdot 12x \neq 0$$

$$4(x-2) + 3x = 7x - 8$$

$$4x - 8 + 3x = 7x - 8$$

$$7x - 8 = 7x - 8 \qquad / \ +8 - 7x$$

$$0 = 0 \quad \textbf{identity}$$

We have used equivalent transformations, and our equation reduces to an identity, which means that every real number for which the original equation is defined (i.e. the expressions in the equation have sense) is a solution to this equation. This means the solutionset is $S = \mathbb{R} \setminus \{0\}$.

- Solve the following equation over the real numbers

$$(x+3) \cdot (3x - 12) = 0.$$

Solution. This equation is clearly not a linear equation. However, using properties of real numbers it can be reduced to two linear equations. Indeed, by Theorem 1.4 a product may be 0 only if one of its factors equals 0. By this, our equation can be replaced by the below two linear equations:

$$x + 3 = 0 \qquad \text{or} \qquad 3x - 12 = 0.$$

We solve both equations separately:

$$x + 3 = 0 \qquad / \ +(-3)$$

$$x = -3$$

and

$$\begin{aligned} 3x - 12 &= 0 && / \ +12 \\ 3x &= 12 && / \ :3 \\ x &= 4 \end{aligned}$$

Now we check the solutions of both equations above, by substituting them into the original equation:

$$(-3+3)\cdot(3(-3)-12) = 0 \ \text{ true},$$

and

$$(4+3)\cdot(3\cdot 4-12) = 0 \ \text{ true}.$$

So the set of solution of our original equation is $S = \{-3, 4\}$.

- Solve the following equation in $x \in \mathbb{R}$:

$$(2x-1)(x+1) - (x+2)^2 + 1 = (x+1)(x-1)$$

Solution. This equation again seems not to be a linear equation. However, when we simplify the expressions in our equation, we shall see that it reduces to a linear equation:

$$\begin{aligned} &2x^2 - x + 2x - 1 - x^2 - 4x - 4 + 1 = x^2 - 1 \\ &x^2 - 3x - 4 = x^2 - 1 && /x^2 + 4 \\ &-3x = 3 && / \ :3 \\ &x = -1. \end{aligned}$$

We check the solution $x = -1$:

$$2\cdot(-1)^2 - (-1) + 2(-1) - 1 - (-1)^2 - 4(-1) - 4 + 1 = (-1)^2 - 1 \quad \text{true}$$

so we get the set of solutions

$$S = \{-1\}.$$

- Solve the following equation in $x \in \mathbb{Z}$:

$$3x - 4 = 0 \qquad\qquad x \in \mathbb{Z}.$$

Solution. We follow the general strategy:

$$\begin{aligned} 3x - 4 &= 0 && / \ +4 \\ 3x &= 4 && / \ :3 \\ x &= \frac{4}{3} \notin \mathbb{Z} \\ S &= \emptyset \end{aligned}$$

Although formally the equation is true for $x = \frac{4}{3}$, this is not a solution, since in this problem we have to search only for integer solutions, i.e. x must be an element of $\mathbb{Z}$. Since the only possible solution does not fulfill this requirement the original equation has solution set $S = \emptyset$.

Exercise 3.1. Solve the following linear equations:

a) $4x - 3 = 2x + 5$ in $x \in \mathbb{R}$

b) $6x - 5 = 4x + 3$ in $x \in \mathbb{R}$

c) $3x - 1 = x + 5$ in $x \in \mathbb{N}$

d) $4x - 1 = 3(x + 1)$ in $x \in \mathbb{R}$

e) $5x - 2 = 4(x - 2) + x + 2$ in $x \in \mathbb{Q}$

f) $4(x - 2) - x = 3(x + 1)$ in $x \in \mathbb{Q}$

g) $4(x - 3) - 3(1 - x) = 3x + 1$ in $x \in \mathbb{Z}$

h) $4(x - 2) - 2(1 - x) = 3(2 - x) + 1$ in $x \in \mathbb{Z}$

i $4(x - 2) - 2(1 - x) = 3(2 - x) + 1$ in $x \in \mathbb{Q}$

j) $\dfrac{x+8}{6} - \dfrac{2x-1}{5} = 2$ in $x \in \mathbb{R}$

k) $\dfrac{x-2}{3} - \dfrac{2x-1}{4} = \dfrac{x+1}{6} - x$ in $x \in \mathbb{R}$

l) $(x - 2)^2 + 2(x + 3)(x - 3) = 3(x + 1)^2 + 3$ in $x \in \mathbb{Z}$

m) $(x - 3)^2 + 2(x + 1)(x - 1) = 3(x - 1)^2 + 15$ in $x \in \mathbb{R}$

n) $\dfrac{x-5}{2} - \dfrac{3x-1}{4} = \dfrac{2x+2}{3} - 3$ in $x \in \mathbb{Q}$

o) $12 - (x - 5) = 20 - (9 - x)$ in $x \in \mathbb{N}$

p) $15(x + 18) = 2(x + 1) + 3(x - 1)$ in $x \in \mathbb{Q}$

q) $2x - \dfrac{3}{5}x = \left(\dfrac{3}{2}x - \dfrac{1}{2}\right) + \left(2 - \dfrac{2}{5}x\right)$ in $x \in \mathbb{R}$

r) $\dfrac{x+3}{4x} - \dfrac{1}{2} = \dfrac{x-1}{12x} + \dfrac{1}{12}$ in $x \in \mathbb{R}$

s) $(x + 3)(2x - 5) = 0$ in $x \in \mathbb{R}$

t) $(x - 4)(5x - 3) = 0$ in $x \in \mathbb{Z}$

u) $(x + 2)(4x - 2)(2x + 6) = 0$ in $x \in \mathbb{R}$

3.3 Quadratic equations

Definition 3.7. An equation is called a **quadratic equation** (or **an equation of degree two**) if it is equivalent to an equation of the form

$$ax^2 + bx + c = 0 \quad \text{where } a, b, c \text{ are constants, and } a \neq 0.$$

The quantity $\Delta := b^2 - 4ac$ is called the **discriminant** of the above equation.

Theorem 3.8. (The "almighty formula")
Let a, b, c be real numbers with $a \neq 0$. Consider the equation

$$ax^2 + bx + c = 0 \quad \textit{where } a, b, c \textit{ are constants, and } a \neq 0. \tag{3.1}$$

Then, depending on the sign of the discriminant $\Delta := b^2 - 4ac$ of the equation we have the following possibilities:

1). **if** $\Delta < 0$ *then equation (3.1)* **has no real solution***;*

2). **if** $\Delta = 0$ *then equation (3.1)* **has two coinciding real solutions***, namely*

$$x_1 = x_2 = \frac{-b}{2a}$$

(Sometimes we simply say that equation (3.1) **has one real solutions***, however this is a somewhat clumsy phrasing);*

3). **if** $\Delta > 0$ *then equation (3.1)* **has two distinct real solutions***, namely*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. To solve equation (3.1) first we do the following equivalent transformations:

$$
\begin{aligned}
&ax^2 + bx + c = 0\\
&a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0\\
&a\left(\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) = 0\\
&a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} = 0\\
&a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a}\\
&\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}
\end{aligned}
$$

Here we have to split the proof according to the sign of $\Delta = b^2 - 4ac$.

1). If $\Delta < 0$ then the right hand side of the last equation is negative, meanwhile the left hand side $\left(x + \frac{b}{2a}\right)^2$ is non-negative (indeed, it is the square of a real number). This is clearly a contradiction, sine a non-negative number is never equal to a negative one.

2). If $\Delta = 0$ then the right hand side of the last equation is 0 and we get

$$\left(x + \frac{b}{2a}\right)^2 = 0$$

which clearly gives

$$x_1 = x_2 = -\frac{b}{2a}.$$

3). If $\Delta > 0$ then $\frac{b^2-4ac}{4a^2} > 0$ and by taking square root of both sides we get

$$\left|x + \frac{b}{2a}\right| = \frac{\sqrt{b^2-4ac}}{2|a|},$$

which in turn gives

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and we get the desired result

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

□

General strategy for solving quadratic equations

1). We transform the equation by equivalent transformations to the form $ax^2 + bx + c = 0$.

2). We compute the discriminant of the resulting equation and we decide if the equation has solutions or not.

3). We use the almighty formula to compute the solutions.

Example. Solve the following quadratic equations:

1). $x^2 - 3x + 2 = 0$

The coefficients are $a = 1, b = -3, c = 2$. The discriminant is

$$\Delta = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1 > 0.$$

Since Δ is positive the equation has two solutions, which can be computed by the "almighty formula":

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-3) \pm \sqrt{1}}{2 \cdot 1} = \frac{3 \pm 1}{2} = \begin{cases} \frac{3+1}{2} = 2 \\ \frac{3-1}{2} = 1 \end{cases}$$

So the set of solutions of the above equation is $S := \{1, 2\}$

2). $x^2 + 4x + 4 = 0$

The coefficients are $a = 1, b = 4, c = 4$. The discriminant is

$$\Delta = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 4 = 16 - 16 = 0.$$

Since $\Delta = 0$ the equation has two coinciding solutions (or we may say that it has only one solution), which can be computed by the "almighty formula":

$$x_1 = x_2 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4) \pm \sqrt{0}}{2 \cdot 1} = -2.$$

So the set of solutions of the above equation is $S := \{-2\}$

3). $x^2 + 2 = 0$

The coefficients are $a = 1, b = 0, c = 2$. The discriminant is

$$\Delta = b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot 2 = 0 - 8 = -8 < 0.$$

Since $\Delta < 0$ the equation has no real solution, i.e. the set of solutions of the above equation is $S := \emptyset$.

Exercise 3.2. Solve the following quadratic equations in $x \in \mathbb{R}$:

a) $x^2 - 1 = 0$

b) $x^2 - 25 = 0$

c) $x^2 - 2 = 0$

d) $x^2 + 3 = 0$

e) $x^2 - 3x = 0$

f) $\mathbf{x}^2 + 5x = 0$

g) $x^2 - 2x + 1 = 0$

h) $x^2 - 4x + 3 = 0$

i) $-x^2 + 5x - 4 = 0$

j) $x^2 + 7x + 12 = 0$

k) $-x^2 - 2x + 3 = 0$

l) $x^2 + 5x - 6 = 0$

m) $x^2 - 5x - 6 = 0$

n) $-x^2 - 6x - 9 = 0$

o) $x^2 + x + 1 = 0$

p) $-x^2 + x - 3 = 0$

q) $2x^2 - 3x + 1 = 0$

r) $6x^2 - x - 2 = 0$

s) $15x^2 - 11x - 12 = 0$

t) $6x^2 + 27x + 30 = 0$

u) $x^2 + x - 4 = 0$

v) $x2 - x + 4 = 0$

w) $3x^2 - 2x + 1 = 0$

x) $4x^2 - 4x + 1 = 0$

y) $9x^2 + 12x + 4 = 0$

z) $4x^2 - 28x + 49 = 0$

Exercise 3.3. Solve the following quadratic equations in $x \in \mathbb{R}$:

a) $2(x+1)^2 - 5(x+3) + 7 = 0$

b) $(x+3)^2 - 2(x+1)(x+2) = 0$

c) $(x+2)^3 - (x+1)(x^2+2) - 4(x+1)(x-1) - 3x + 2 = 0$

d) $(x-3)^2 - 2(x+1)(x+2) + 10x + 10 = 0$

e) $(x+4)^2 - 2(x-1)(x+2) + 10x - 84 = 0$

f) $(x-5)^2 + 2(x+3)(x+2) + 3x - 34 = 0$

g) $\dfrac{2}{x^2-4} - \dfrac{1}{x(x-2)} + \dfrac{x-4}{x(x+2)}$

h) $\dfrac{1}{x-1} + \dfrac{2}{x-2} + \dfrac{3}{x-3} = \dfrac{6}{x+6}$

i) $\dfrac{1}{4x+8} = \dfrac{20x+1}{4x^2-16} - \dfrac{7-5x}{x^2-4x+4}$

j) $\dfrac{1}{x^2-x} + \dfrac{1}{x^2-3x+2} - \dfrac{1}{x-2} = 0$

Chapter 4

Inequalities

4.1 Introduction to inequalities

Definition 4.1. An **inequality** is a statement that an algebraic expression is *smaller* then (or *larger than*, or *smaller or equal to* or *larger or equal to*) another algebraic expression.

Example. By the above definition the statement

$$1 < 2$$

is an inequality, however, in algebra we are more interested in inequalities which contain variables (i.e. letters). The following statements are also examples of inequalities:

$$3x > x + 2 \qquad x^2 + y^2 \leq 1$$

$$\sqrt{x^2 + 1} \leq 3 \qquad |x| \geq 3$$

Example. Similarly to the case of equations, here are three examples of inequalities which look completely similar:

1). $3x + 4 - 3(x + 3) + 5 < 1$,

2). $3x + 4 - 3(x + 3) + 5 < 0$,

3). $3x + 4 - 2(x + 3) + 2 < 0$.

However, if we simplify the left hand sides of the above equations we get

1). $0 < 1$,

2). $0 < 0$,

3). $x < 0$.

So we can see that we may have three completely different situations:

1). the inequality is **true for all possible values** of x,

2). the inequality is **false for all possible values** of x,

3). the inequality is true for a part of the possible values of x, so the **truth of the inequality depends on the value of** x.

The above example justifies to give separate names for such inequalities, as we shall do it in the definitions below. The names given to the different types of inequalities are the same like names of the corresponding cases of equations.

Definition 4.2. An inequality which is always false, independently of the value of the variables involved, is called a **contradiction**.

Definition 4.3. An inequality which is always true, independently of the value of the variables involved, is called an **identity**.

Definition 4.4. An inequality whose truth depends on the value of the variables involved is called a **conditional inequality**. To **solve** a conditional inequality means to find the set of those values of the variables for which the equation is true. These values are called **solutions** of the inequality.

Definition 4.5. Two inequalities that have exactly the same set of solutions are called **equivalent inequalities**. Transformations that modify an inequality to an equivalent inequality are called **equivalent transformations** of the inequality.

Equivalent Transformations of Inequalities

1). Equivalent transformation of the algebraic expression on the left hand side

2). Equivalent transformation of the algebraic expression on the right hand side

3). Adding the same number or algebraic expression to both sides

$$a < b \iff a + c < b + c \qquad a > b \iff a + c > b + c$$
$$a \leq b \iff a + c \leq b + c \qquad a \geq b \iff a + c \geq b + c$$

4). Subtracting the same number or algebraic expression from both sides

$$a < b \iff a - c < b - c \qquad a > b \iff a - c > b - c$$
$$a \leq b \iff a - c \leq b - c \qquad a \geq b \iff a - c \geq b - c$$

5). Multiplying both sides by the same positive number or positive algebraic expression

$$a < b \iff ac < bc \text{ with } c > 0 \qquad a > b \iff ac > bc \text{ with } c > 0$$
$$a \leq b \iff ac \leq bc \text{ with } c > 0 \qquad a \geq b \iff ac \geq bc \text{ with } c > 0$$

6). Multiplying both sides by the same negative number or negative algebraic expression and changing the direction of the inequality sign

$$a < b \iff ac > bc \text{ with } c < 0 \qquad a > b \iff ac < bc \text{ with } c < 0$$

$$a \leq b \iff ac \geq bc \text{ with } c < 0 \qquad a \geq b \iff ac \leq bc \text{ with } c < 0$$

7). Dividing both sides by the same positive number or positive algebraic expression

$$a < b \iff \frac{a}{c} < \frac{b}{c} \text{ with } c < 0 \qquad a > b \iff \frac{a}{c} > \frac{b}{c} \text{ with } c < 0$$

$$a \leq b \iff \frac{a}{c} \leq \frac{b}{c} \text{ with } c < 0 \qquad a \geq b \iff \frac{a}{c} \geq \frac{b}{c} \text{ with } c < 0$$

8). Dividing both sides by the same negative number or negative algebraic expression and changing the direction of the inequality sign

$$a < b \iff \frac{a}{c} > \frac{b}{c} \text{ with } c < 0 \qquad a > b \iff \frac{a}{c} < \frac{b}{c} \text{ with } c < 0$$

$$a \leq b \iff \frac{a}{c} \geq \frac{b}{c} \text{ with } c < 0 \qquad a \geq b \iff \frac{a}{c} \leq \frac{b}{c} \text{ with } c < 0$$

Convention. When solving inequalities, our goal is to give the set of all solutions, i.e. that simplifying the inequality to a very simple equivalent inequality like $x < 2$ is not a complete solution of the problem, since we are expected to solve the resulting simple inequality and give the result in the form of an interval, like $x \in]-\infty, 2[$ in the case of the above example.

4.2 Linear inequalities

Definition 4.6. An inequality is called a **linear inequality if it is equivalent to one of the following inequalities, where** $a, b \in \mathbb{R}$ **and** $a \neq 0$

$$ax - b < 0 \qquad ax - b > 0$$

$$ax - b \leq 0 \qquad ax - b \geq 0$$

Method of solving linear equations

1). Using equivalent transformations of the inequality transform it to one of the following basic forms:

$ax - b < 0$	$ax - b > 0$
$ax - b \leq 0$	$ax - b \geq 0$

2). Add b to both sides of the resulting basic inequality

3). If $a \neq 0$ then divide the inequality by a, and if a is negative then change the direction of the inequality sign

4). Decide if the inequality is a contradiction or an identity, and if not, then determine the set of solutions.

Example. Solve the following inequality in $x \in \mathbb{R}$:

$$\frac{4x-3}{5} - \frac{x-4}{2} < 2.$$

To solve the above inequality we use equivalent transformations of it as follows

$$\begin{aligned}
&\frac{4x-3}{5} - \frac{x-4}{2} < 2 && / \cdot 10 > 0 \\
&2(4x-3) - 5(x-4) < 20 && \\
&8x - 6 - 5x + 20 < 20 && \\
&3x + 14 < 20 && / - 14 \\
&3x < 6 && / : 3 > 0 \\
&x < 2 &&
\end{aligned}$$

so the set of solutions is $S :=]-\infty, 2[$

Exercise 4.1. Solve the following linear inequalities:

a) $9x + 8 \leq 15 + 7x$

b) $5(x-1) + 7 < 7(x+2) - 4$

c) $4x + 12 \leq 10 - 6(x-2)$

d) $5(x+1) - 9(x+3) > -6(x+2)$

e) $\dfrac{2x-4}{-5} > 0$

f) $\dfrac{3x+2}{4}$

g) $-2x + 6 \leq \dfrac{1}{2}x + 1$

h) $\dfrac{1}{2} - 3x < \dfrac{2}{3}x - 5$

i) $3x - \dfrac{1}{6} > 2 - \dfrac{2x}{3}$

j) $\dfrac{x}{2} + \dfrac{3x}{4} > \dfrac{5x}{6} + 5$

k) $\dfrac{4x}{3} + \dfrac{x}{6} < \dfrac{3x}{4} + 3$

l) $\dfrac{x}{3} + \dfrac{2x}{5} > \dfrac{7x}{10} + 1$

m) $\dfrac{7x}{8} - 5 \leq \dfrac{9x}{10} - 8$

n) $\dfrac{5-z}{8} > \dfrac{18-z}{12}$

o) $\dfrac{5x-4}{2} < \dfrac{16x+1}{7}$

p) $\dfrac{2x+4}{2} - 2 > 3(x+2)$

q) $\dfrac{5}{3}x + 5(4-x) < 2(4-x)$

r) $\dfrac{7-x}{2} + 4 < \dfrac{3+4x}{5} + 3$

s) $\dfrac{3x-5}{2} - 1 \leq \dfrac{x-11}{5}$

t) $2(x-2) \leq 2x - 7$

u) $3x - 5 \leq 7 + 3(x-1)$

v) $5x - 3(x-2) \geq 2(x+3)$

w) $5x - 3(x-2) \leq 2(x+3)$

x) $6x - 3(x-3) < 2(x+4) + x + 1$

y) $6x - 3(x-3) > 2(x+4) + x + 1$

z) $11x - 7(x-3) < 2(x+4) + 2x + 1$

4.3 Table of signs

A special case of the inequalities is when the left hand side is a rational algebraic expression in a single variable and the right hand side is 0. In this case in fact we have to determine the sign of the rational algebraic expression appearing on the left hand side of the inequality. The result can be summarized in a so-called **table of signs**.

The table of signs is table in which:

- in the first row we list in increasing order all those values of the variable for which the expression is 0 or has no sense;
- the last row is built such that we put 0 below the values of the variable represented in the first row, if the value of the expression at these values is 0, and
- we put a vertical line below the values of the variable represented in the first row, if the expression has no sense at these values of the variable, and
- below the intervals determined by the consecutive values represented in the first row we write + signs if the expression is positive over this interval, and − signs if the expression is negative over this interval;
- in the optional intermediate rows we represent the sign of factors of the numerator or denominator of the expression, which rows help us to determine the sign of the whole expression.

Remark. The above described table of signs is used for analyzing the sign of more complicated expressions, however there are several variants of the idea of table of signs, which are used to different tasks in algebra. Later we shall see how to simplify the solution of equations containing absolute values with the help of a table of signs.

In the sequel we shall present basic tables of signs which completely describe the sign of linear and quadratic expressions, and then we shall use these rules to solve more complicated exercises.

4.3.1 The sign of linear expressions

In this section we consider the expression $ax+b$ with $a, b \in \mathbb{R}$ and $a \neq 0$, and we prove a theorem which summarizes all possible cases for determining the sign of this expression, depending on the values of x.

Theorem 4.7. *The below simple table of signs describes the sign of the expression $ax+b$ where $a, b \in \mathbb{R}$ and $a \neq 0$.*

	$-\infty$		$\frac{-b}{a}$		∞
$ax+b$		*sign opposite to the sign of a*	*0*	*sign of a*	*sign of a*

Example. Using a table of signs describe the sign of the expression $2x-6$.

Solution. Clearly $2x-6$ takes zero if and only if

$$2x - 6 = 0$$

which means that $x = 3$. So we have to write 3 in the first row of our table, and then below the 3 we put 0 in the second row. We fill by $-$ (the sign opposite to the sign of 3) the left half of the second row, and by $+$ (the sign of 3) the right half of the second row.

	$-\infty$	3	∞
$2x-6$	- -	0	+ + + + + + + + + + + +

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $-2x-10$.

Solution. Clearly $-2x-10$ takes zero if and only if

$$-2x - 10 = 0$$

which means that $x = -5$. So we have to write -5 in the first row of our table, and then below the -5 we put 0 in the second row. We fill by $+$ (the sign opposite to the sign of -5) the left half of the second row, and by $-$ (the sign of -5) the right half of the second row.

	$-\infty$	-5	∞
$-2x-10$	+ + + + + + + + + + + +	0	- - - - - - - - - - - - - - - - - - - -

Now it is easy to read the result from the above table.

4.3.2 The sign of quadratic expressions

In this section we consider the expression ax^2+bx+c with $a, b, c \in \mathbb{R}$ and $a \neq 0$, and we prove a theorem which summarizes all possible cases for determining the sign of this expression, depending on the values of x. We have to split the discussion in three cases depending on the sign of the discriminant.

Theorem 4.8. *Let $\Delta := b^2-4ac$ be the discriminant of the polynomial ax^2+bx+c. The below three simple tables of signs describe the sign of the expression ax^2+bx+c where $a, b, c \in \mathbb{R}$ and $a \neq 0$.*

1). If the discriminant is negative, i.e. $\Delta < 0$, then the equation $ax^2+bx+c = 0$ has no real zeros, and we have

	$-\infty$			∞
ax^2+bx+c		*sign of* a	*sign of* a	*sign of* a

2). If the discriminant is zero, i.e. $\Delta = 0$, then the equation $ax^2+bx+c = 0$ has two coinciding zeros, $x_1 = x_2$, and we have

	$-\infty$			x_1			∞
ax^2+bx+c		*sign of a*	*sign of a*	*0*	*sign of a*	*sign of a*	

3). If the discriminant is positive, i.e. $\Delta > 0$*, then the equation* $ax^2+bx+c=0$ *has two distinct zeros,* $x_1 < x_2$*, and we have*

	$-\infty$		x_1		x_2		∞
ax^2+bx+c		*sign of a*	*0*	*sign opposite to sign of a*	*0*	*sign of a*	

Example. Using a table of signs describe the sign of the expression x^2-6x+5.

Solution. Clearly x^2-6x+5 takes zero if and only if

$$x^2-6x+5=0$$

which means that $x_1=1$ or $x_2=5$. So we have to write 1 and 5 in the first row of our table, and then below them we put 0 in the second row. We fill by $-$ (the sign opposite to the sign of the leading coefficient $a=+1$) the the second row between the two zeros, and by $+$ (the sign of $a=+1$) the two sides of the second row.

	$-\infty$		1		5		∞
x^2-6x+5		+ + + + + + + + +	0	- - - - - - - - - - - -	0	+ + + + + + +	+

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $-2x^2+6x-4$.

Solution. Clearly $-2x^2+6x-4$ takes zero if and only if

$$-2x^2+6x-4=0$$

which means that $x_1=1$ or $x_2=2$. So we have to write 1 and 2 in the first row of our table, and then below them we put 0 in the second row. We fill by $+$ (the sign opposite to the sign of the leading coefficient $a=-2$) the the second row between the two zeros, and by $-$ (the sign of $a=-2$) the two sides of the second row.

	$-\infty$		1		2		∞
$-2x^2+6x-4$	- - - - - - - - - - - - - - - -		0	+ + + + + + +	0	- - - - - - - - - - - - - -	

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $2x^2+8x+8$.

Solution. Clearly $2x^2+8x+8$ takes zero if and only if

$$2x^2+8x+8=0$$

which means that $x_1=x_2=-2$. So we have to write -2 in the first row of our table, and then below the -2 we put 0 in the second row. Finally, we fill by $+$ (the sign of the leading coefficient $a=2$) the the second row around the 0.

	$-\infty$		-2		∞
$2x^2+8x+8$	+ + + + + + + + + + + + +		0	+ + + + + + + + + + + + +	

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $-x^2+6x-9$.

Solution. Clearly $-x^2+6x-9$ takes zero if and only if

$$-x^2+6x-9=0$$

which means that $x_1=x_2=3$. So we have to write 3 in the first row of our table, and then below the 3 we put 0 in the second row. Finally, we fill by $-$ (the sign of the leading coefficient $a=-1$) the the second row around the 0.

	$-\infty$		3		∞
$-x^2+6x-9$	- - - - - - - - - - - - - - - - - - - -		0		- - - - - - - - - - - - - - - - - - - -

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $2x^2+x+2$.

Solution. Clearly $2x^2+x+2$ takes zero if and only if

$$2x^2+x+2=0,$$

however, computing the discriminant of this equation we get $\Delta=1^2-4\cdot 2\cdot 2=-15<0$, so this equation has no real solutions. Thus we do not write anything in the first row of our table (except for $-\infty$ and ∞ at the two ends), and we fill by $+$ (the sign of the leading coefficient $a=2$) the the second row.

	$-\infty$	∞
$2x^2+x+2$	+ +	

Now it is easy to read the result from the above table.

Example. Using a table of signs describe the sign of the expression $-x^2+x-1$.

Solution. Clearly $-x^2+x-1$ takes zero if and only if

$$-x^2+x-1=0,$$

however, computing the discriminant of this equation we get $\Delta = 1^2 - 4 \cdot (-1) \cdot (-1) = -3 < 0$, so this equation has no real solutions. Thus we do not write anything in the first row of our table (except for $-\infty$ and ∞ at the two ends), and we fill by $-$ (the sign of the leading coefficient $a = -1$) the the second row.

	$-\infty$	∞
$-x^2 + x - 1$	- -	

Now it is easy to read the result from the above table.

4.4 Quadratic inequalities

Strategy of solving quadratic inequalities

1). First we transform the inequality to the form $ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

2). We use the tables of signs described in Theorem 4.8 to analyze the sign of the expression $ax^2 + bx + c$.

3). We read the result from the above-mentioned table of signs.

Example. Solve the following inequality in real values of x

$$-x^2 + 5x - 6 \leq 0.$$

Solution. We build a table of sings to describe the sign of the expression $-x^2+5x-6$ Clearly $-x^2 + 5x - 6$ takes zero if and only if

$$-x^2 + 5x - 6 = 0$$

which means that $x_1 = 2$ or $x_2 = 3$. So we have to write 2 and 3 in the first row of our table, and then below them we put 0 in the second row. Finally, we fill by $+$ (the sign opposite to the sign of the leading coefficient $a = -1$) the second row between the two zeros, and by $-$ (the sign of $a = -1$) the two sides of the second row.

	$-\infty$	2		3		∞
$-x^2 + 5x - 6$	- - - - - - - - - - - - - - - -	0	+ + + + + + +	0	- - - - - - - - - - - - - -	

So the solution to our inequality is

$$x \in \left]-\infty, 2\right] \cup \left[3, \infty\right[.$$

Example. Solve the following inequality in real values of x

$$-x^2 + 5x - 6 < 0.$$

Solution. We build the very same table of sings to describe the sign of the expression $-x^2 + 5x - 6$, as in the previous example, and now we read the result

$$x \in \left]-\infty, 2\right[\cup \left]3, \infty\right[.$$

4.4.1 Graphical approach of quadratic inequalities

We reconsider the previos example.

Example. Solve the following inequality in real values of x

$$-x^2 + 5x - 6 < 0.$$

Solution. Completing the square we have

$$-x^2 + 5x - 6 = -(x - 2.5)^2 + \frac{1}{4}.$$

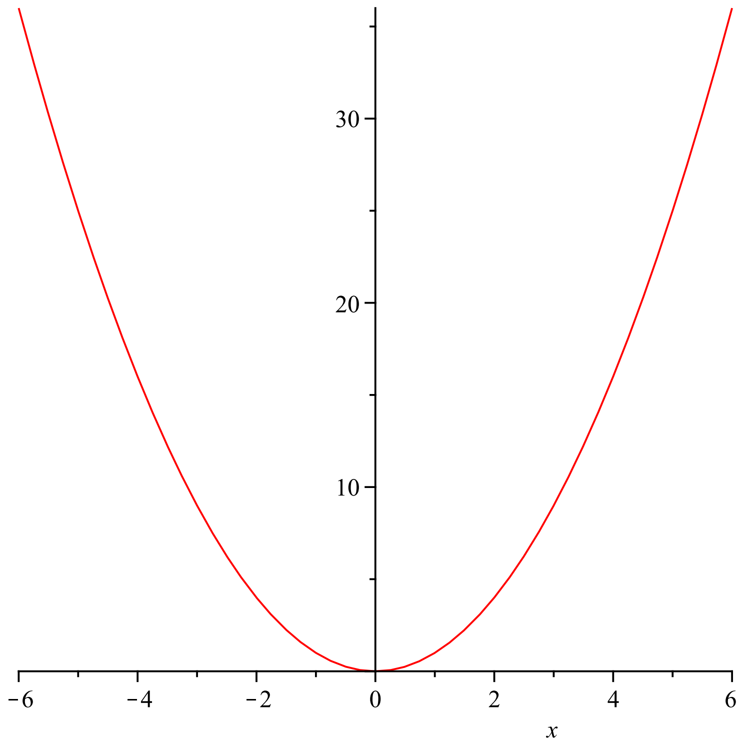

Figure 4.1: Graph of the function x^2

We have to give the graph of $f(x) = -x^2 + 5x - 6 = -(x - 2.5)^2 + \frac{1}{4}$. First we get the graph of $f_1(x) = x^2$, see Figure 4.1. As the second step we yield the graph of $f_2(x) = (x - 2.5)^2$ on the Figure 4.2. Now one can easily obtain the graph of $f_3(x) = -(x - 2.5)^2$, see Figure 4.3. Finally, Figure 4.4 shows tha graph of the original quadratic function. One can see that the set of solutions to the inequality

$$-x^2 + 5x - 6 < 0$$

is $x < 2$ or $x > 3$.

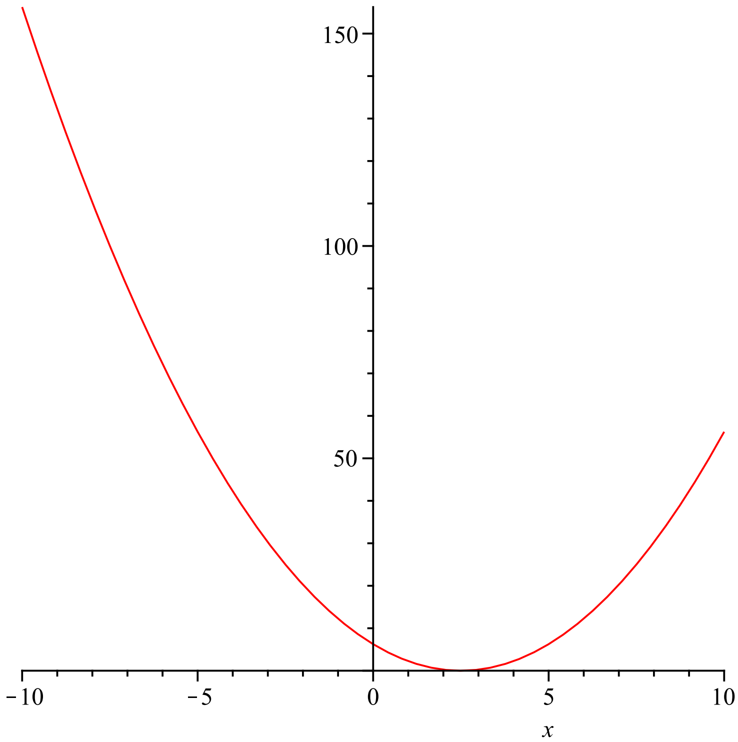

Figure 4.2: Graph of the function $(x - 2.5)^2$

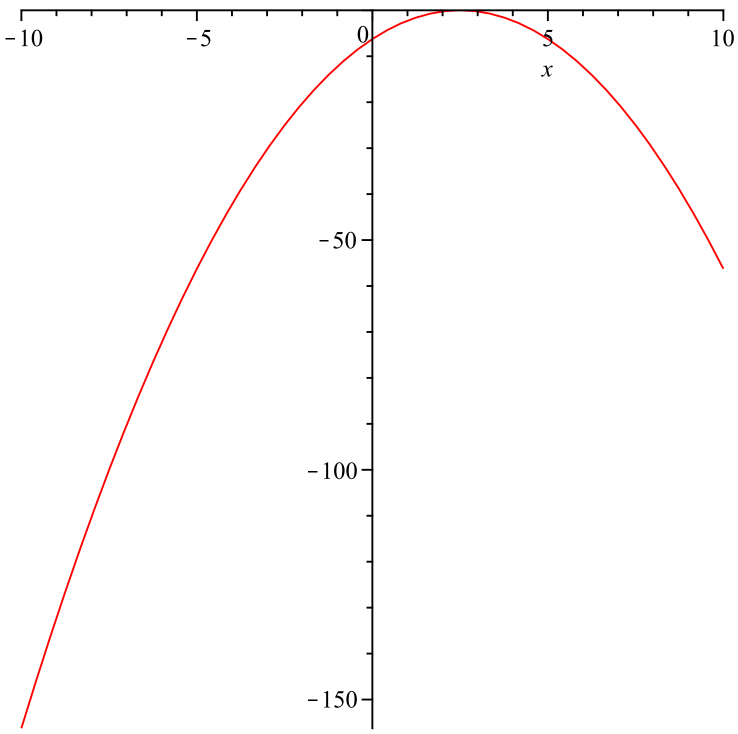

Figure 4.3: Graph of the function $-(x-2.5)^2$

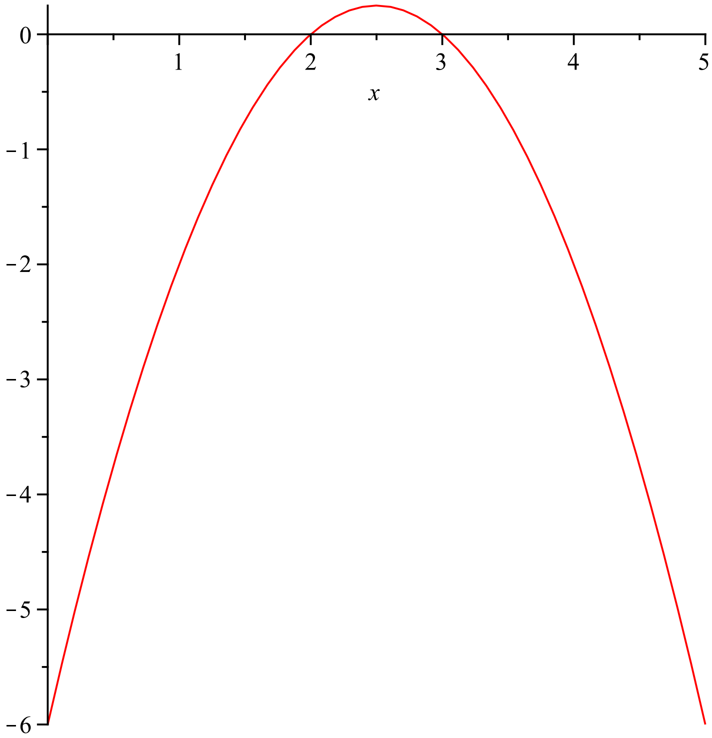

Figure 4.4: Graph of the function $-(x-2.5)^2+\frac{1}{4}$

Exercise 4.2. Solve the following inequality in real values of x:

a) $x^2 - 4x + 3 > 0$	b) $2x^2 + 6x + 4 \geq 0$
c) $-2x^2 - 10x - 6 > 0$	d) $x^2 + x - 6 \leq 0$
e) $x^2 - 4 \geq 0$	f) $-x^2 + 9 > 0$
g) $x^2 - 2x + 1 \geq 0$	h) $-x^2 + 6x - 9 \geq 0$
i) $2x^2 + 8x + 8 > 0$	j) $-x^2 - 8x - 10 < 0$
k) $x^2 - x + 1 < 0$	l) $x^2 - x + 1 \geq 0$
m) $x^2 - x + 1 > 0$	n) $-x^2 + x - 3 \leq 0$
o) $-x^2 - 3 \geq 0$	p) $-x^2 - x - 1 < 0$

4.5 Solving inequalities using table of signs

Example. Solve the following inequality in $x \in \mathbb{R}$:

$$\frac{(x-2)(3-x)(x^2-x+1)}{(x^2+4x+3)(-x+1)(x+3)} \leq 0$$

Solution. We shall use table of signs. To do that we first solve all factors of the numerator and the denominator for zero:

$x - 2 = 0$	$3 - x = 0$	$x^2 - x + 1 = 0$	$x^2 + 4x + 3$	$-x + 1$	$x + 3$
$x_1 = 2$	$x_2 = 3$	$x \notin \mathbb{R}$	$x_{3,4} = -1; -3$	$x_5 = 1$	$x_6 = -3$

We insert all these solutions in the first row of the table in increasing order. Then we analyze the sign of these factors using separate rows of a joint table of signs, and finally, in the last row we compute the sign of the algebraic expression

$$F := \frac{(x-2)(3-x)(x^2-x+1)}{(x^2+4x+3)(-x+1)(x+3)}$$

composed using these factors. In the last row, below the separating points $x_1, x_2, \ldots$ we put 0 if no factor of the denominator vanishes at that number, and we put a vertical line if the number in question is a zero of a factor of the denominator. The vertical line means that F has no sense, and the 0 means that $F = 0$ at the number which is in the first row above the vertical line, and the 0, respectively.

	$-\infty$	-3		-1		1		2		3	∞
$x-2$	- - -	-	- - -	-	- - -	-	- - -	0	+ + +	+	+ + +
$3-x$	+ + +	+	+ + +	+	+ + +	+	+ + +	+	+ + +	0	- - -
x^2-x+1	- - -	-	- - -	-	- - -	-	- - -	-	- - -	-	- - -
x^2+4x+3	+ + + +	0	- - - - -	0	+ + +	+	+ + +	+	+ + +	+	+ + +
$-x+1$	+ + +	+	+ + +	+	+ + +	0	- - -	-	- - -	-	- - -
$x+3$	- - - - - - -	0	+ + +	+	+ + +	+	+ + +	+	+ + +	+	+ + +
F	- - - - - - -	∣	- - - - -	∣	+ + +	∣	- - - - -	0	+ + +	0	- - - - - - - -

Finally we have to read the result from the table of signs: $F \leq 0$ for those values of x which are from intervals (represented in the first row) being above $-$ signs of the last row. The intervals are closed if the endpoint of the interval is above a zero of the last row. So the solution of our example is:

$$x \in \,]-\infty, -3[\, \cup \,]-3, -1[\, \cup \,]1, 2] \cup [3, \infty[$$

□

Exercise 4.3. Solve the following inequalities in $x \in \mathbb{R}$:

a) $\dfrac{(x-5)(3+x)(x^2-3x+2)}{(x^2+4x+3)(-x+4)(x-7)} > 0$

b) $\dfrac{(x+3)(7-x)(-x^2-x-1)}{(x^2-4x+3)(-x+1)(2x+10)} \geq 0$

c) $\dfrac{(-4x-2)(8-2x)(x^2-2x+1)}{(x^2+4x+5)(-x-1)(2x+6)} \leq 0$

d) $\dfrac{(3x-9)(3+3x)(-x^2+x-1)}{(x^2+4x+4)(-x+1)(2x+6)} > 0$

e) $\dfrac{(-x+5)(3+x)(x^2-5x-6)}{(2x^2-10x+12)(x+8)(-x+4)} \leq 0$

f) $\dfrac{(-x-2)(5+x)(-x^2-x+2)}{(x^2+7x+12)(-2x+8)(x+7)} \geq 0$

Chapter 5

Equations II

5.1 Equations containing absolute values

When we have to solve equations containing absolute values in most case we have to "get rid" of the absolute values involved, and solve the resulting equations. The way we can "get rid" of the absolute values is to use the formula

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) \leq 0. \end{cases}$$

However, this means that we have to split the solution into subcases, depending on the sign of the expressions appearing in absolute value. For instance, in the case of equation

$$|x-1| + |x-3| = 10$$

for the first sight it seems that we have to distinguish four cases, according to the sign of $x-1$ and $x-3$:

$$\begin{cases} x-1 \le 0 \\ x-3 \le 0 \end{cases} \qquad \begin{cases} x-1 \le 0 \\ x-3 \ge 0 \end{cases} \qquad \begin{cases} x-1 \ge 0 \\ x-3 \le 0 \end{cases} \qquad \begin{cases} x-1 \ge 0 \\ x-3 \ge 0. \end{cases}$$

However, a more careful analysis shows that the second system of inequalities has no solutions, so we do not have to consider this case. If the number of expressions appearing in absolute value is larger, then the cases to consider "a priori" is even larger, and also the cases where the solution set of the system of inequalities is the emptyset will increase. This situation can be handled much easier with the help of a table of signs. Let us solve step by step the above equation:

Example. Solve the following equation in $x \in \mathbb{R}$

$$|x-1| + |x-3| = 10$$

Solution. We shall do equivalent transformations. We have to get rid of the absolute values by the formulas

$$|x-1| = \begin{cases} (x-1) & \text{if } x-1 \ge 0 \\ -(x-1) & \text{if } x-1 \le 0. \end{cases} \qquad |x-3| = \begin{cases} (x-3) & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 \le 0. \end{cases}$$

We build a table of signs containing all expressions appearing in absolute value:

$$x-1=0 \qquad x-3=0$$

$$x_1 = 1 \qquad x_2 = 3$$

	$-\infty$	1	3	∞
$x-1$	- - - - - - - - - - - -	0	+ + + + + + + + + + + +	
$x-3$	- -		0	+ + + + + +

Now we split the solution into subcases. We have to consider all intervals appearing in the first row of the table of signs (namely 3 intervals). The easiest way is to take closed ending at each finite endpoint of each interval, this way considering these numbers twice. However, it is more elegant to consider each number only once, and here we shall take care of this. It is very important that a solution of the resulting equation in a subcase is accepted as a solution of the original equation only if the solution is an element of the defining interval of that very case. Indeed, otherwise the absolute values would have been computed by another formula, and that number should be a solution of a different equation.

Case 1. If $x \in]-\infty, 1[:= I_1$ then

$$-(x-1)+(-(x-3)) = 10$$
$$-(x-1)-(x-3) = 10$$
$$-x+1-x+3 = 10$$
$$-2x = 6$$
$$x = -3 \in I_1$$

Since the solution we got is included in the interval defining Case 1, thus the solution set of the first case is

$$S_1 = \{-3\}$$

Case 2. If $x \in [1, 3[:= I_2$ then

$$(x-1)+(-(x-3)) = 10$$
$$(x-1)-(x-3) = 10$$
$$x-1-x+3 = 10$$
$$0 = 8 \quad \text{contradiction}$$

So the solution set of the second case is

$$S_2 = \emptyset$$

Case 3. If $x \in [3, \infty[:= I_3$ then

$$(x-1)+(x-3) = 10$$
$$(x-1)+(x-3) = 10$$
$$x-1+x-3 = 10$$
$$2x = 14$$
$$x = 7 \in I_3$$

So the solution set of the third case is

$$S_3 = \{7\}$$

To get the complete solution set of the equation we have to join the sets of solutions of the above subcases, i.e.

$$S = S_1 \cup S_2 \cup S_3 = \{-3, 7\}$$

Example. Solve the following equation in $x \in \mathbb{R}$:

$$|x+1| + |x-2| + |3-x| = 6.$$

Solution. We shall do equivalent transformations. We have to get rid of the absolute values by the formulas

$$|x+1| = \begin{cases} (x+1) & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 \leq 0. \end{cases}$$

$$|x-2| = \begin{cases} (x-2) & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 \leq 0. \end{cases}$$

$$|3-x| = \begin{cases} (3-x) & \text{if } 3-x \geq 0 \\ -(3-x) & \text{if } 3-x \leq 0. \end{cases}$$

We build a table of signs containing all expressions appearing in absolute value:

$$x+1=0 \qquad x-2=0 \qquad 3-x=0$$

$$x_1=-1 \qquad x_2=2 \qquad x_3=3$$

	$-\infty$		-1		2		3		∞
$x+1$	- - -	- - -	0	+ + +	+ + +	+ + +	+ + +	+ + +	+ + +
$x-2$	- - -	- - -	- - -	- - -	0	+ + +	+ + +	+ + +	+ + +
$3-x$	+ + +	+ + +	+ + +	+ + +	+ + +	+ + +	0	- - -	- - -

We split the solution into subcases. We have to consider all intervals appearing in the first row of the table of signs (namely 4 intervals).

Case 1. If $x \in]-\infty, -1[:= I_1$ then

$$-(x+1)-(x-2)+(3-x)=6$$
$$-x-1-x+2+3-x=6$$
$$-3x=2$$
$$x=-\frac{2}{3}\notin I_1$$

Since the solution we got is not in the interval defining Case 1, thus the solution set of the first case is

$$S_1=\emptyset.$$

Case 2. If $x \in [-1, 2[:= I_2$ then

$$(x+1)-(x-2)+(3-x)=6$$
$$x+1-x+2+3-x=6$$
$$-x=0$$
$$x=0\in I_2$$

So the solution set of the second case is

$$S_2=\{0\}.$$

Case 3. If $x \in [2, 3[:= I_3$ then

$$(x+1)+(x-2)+(3-x)=6$$
$$x=4\notin I_3$$

So the solution set of the third case is

$$S_3=\emptyset$$

Case 4. If $x \in [3, \infty[:= I_4$ then

$$(x+1)+(x-2)-(3-x)=6$$
$$x+1+x-2-3+x=6$$
$$3x=10$$
$$x=\frac{10}{3}\in I_4$$

So the solution set of the third case is

$$S_4=\left\{\frac{10}{3}\right\}$$

To get the complete solution set of the equation we have to join the sets of solutions of the above subcases, i.e.

$$S=S_1\cup S_2\cup S_3\cup S_4=\left\{0,\frac{10}{3}\right\}$$

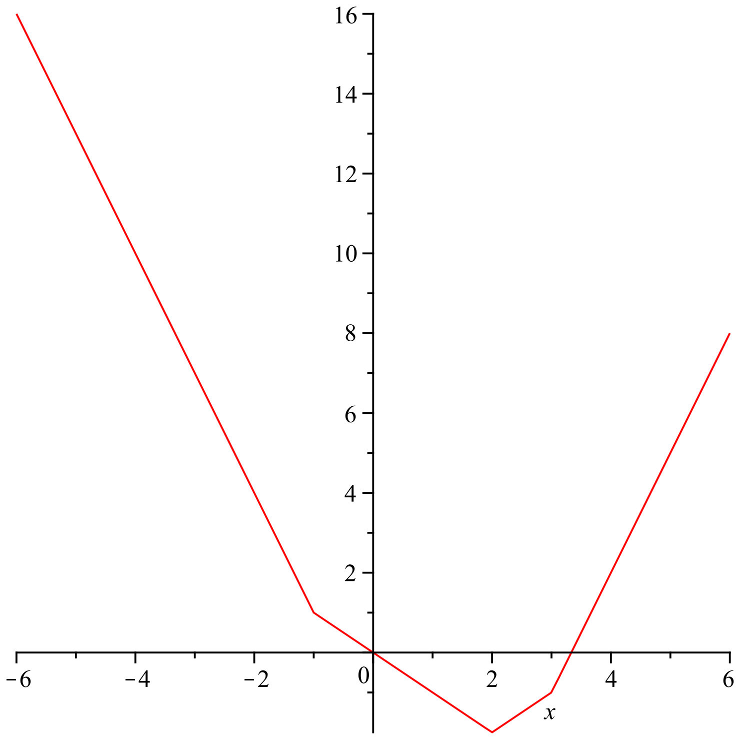

Figure 5.1: Graph of the function $|x+1|+|x-2|+|3-x|-6$

For the graph of $|x+1|+|x-2|+|3-x|-6$ see Figure 5.1

Example. Solve the following equation in $x \in \mathbb{R}$

$$|x-1|+|x+1|=2.$$

Solution. We shall do equivalent transformations. We have to get rid of the absolute values by the formulas

$$|x-1| = \begin{cases} (x-1) & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 \leq 0. \end{cases} \qquad |x+1| = \begin{cases} (x+1) & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1 \leq 0. \end{cases}$$

We build a table of signs containing all expressions appearing in absolute value:

$$x - 1 = 0 \qquad x + 1 = 0$$

$$x_1 = 1 \qquad x_2 = -1$$

	$-\infty$		-1		1		∞
$x-1$	- - -	- - -	- - -	- - -	0	+ + +	+ + +
$x+1$	- - -	- - -	0	+ + +	+ + +	+ + +	+ + +

Now we split the solution into subcases. We have to consider all intervals appearing in the first row of the table of signs (namely 3 intervals).

Case 1. If $x \in \,]-\infty, -1[\, := I_1$ then

$$-(x-1) + (-(x+1)) = 2$$

$$-x + 1 - x - 1 = 2$$

$$-2x = 2$$

$$x = -1 \notin I_1$$

Since the solution we got is not in the interval defining Case 1, thus the solution set of the first case is

$$S_1 = \emptyset.$$

Case 2. If $x \in [-1, 1[\, := I_2$ then

$$-(x-1) + (x+1) = 2$$

$$-x + 1 + x + 1 = 2$$

$$0 = 0 \quad \text{identity}$$

So the solution set of the second case is the whole interval I_2:

$$S_2 = I_2 = [-1, 1[\,.$$

Case 3. If $x \in [1, \infty[\, := I_3$ then

$$(x-1) + (x+1) = 2$$

$$2x = 2$$

$$x = 1 \in I_3$$

So the solution set of the third case

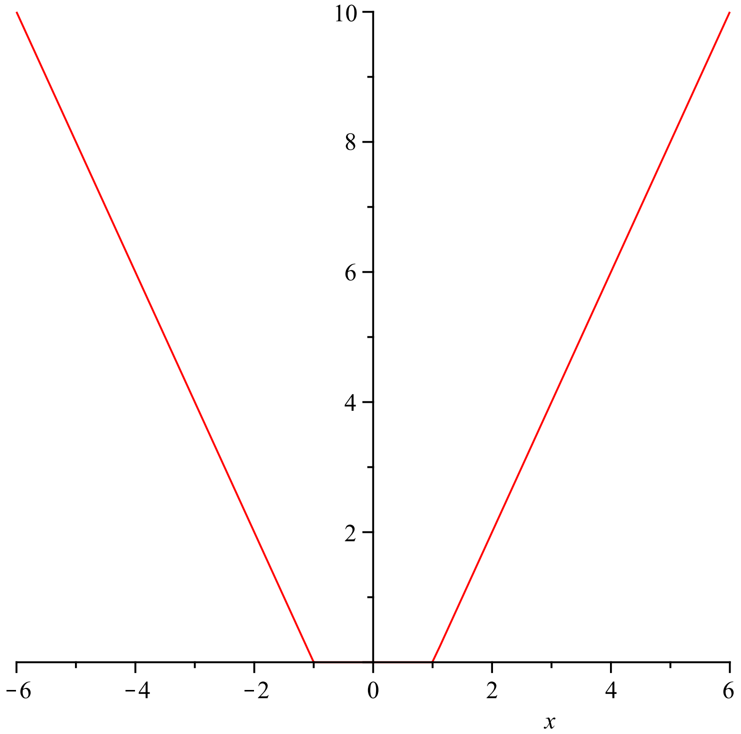

Figure 5.2: Graph of the function $|x-1|+|x+1|-2$

is

$$S_3 = \{1\}$$

To get the complete solution set of the equation we have to join the sets of solutions of the above subcases, i.e.

$$S = S_1 \cup S_2 \cup S_3 = [-1, 1]$$

For the graph of $|x-1|+|x+1|-2$ see Figure 5.2

5.2 Polynomial equations of higher degree

We have a simple procedure to solve linear equations, and the almighty formula allows us to solve easily quadratic equations. To solve polynomial equations of higher degree is much harder. For cubic and quartic equations there exist formulas, but these are very complicated. The present knowledge of the reader of this book makes it possible to solve polynomial equations of higher degree only if the equation is of some special shape, which allows the use of some clever method, to reduce the degree of the equation. In the present section we show the most important types of such equations.

5.2.1 Solving polynomial equations by finding rational roots

In this subsection we will solve polynomial equations of higher degree with integer coefficients by reducing their degree via finding their rational solutions. The following theorem makes possible to find all rational solutions, and if we are lucky to have enough rational solutions, maybe we can completely solve our equation.

Theorem 5.1. (The rational root theorem) *Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, having the form*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If a rational number $\frac{u}{v} \in \mathbb{Q}$ $(u, v \in \mathbb{Z}, v \neq 0)$ in its lowest terms (i.e. with $\gcd(u, v) = 1$) is a solution to the equation

$$P(x) = 0$$

then the numerator u is a divisor of the free term a_0 $(u \mid a_0)$ and the denominator v is a divisor of the leading coefficient a_n $(v \mid a_n)$.

As special cases of the above Theorem 5.1, we get the following corollaries:

Corollary 5.2. (The integer root theorem) *Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, having the form*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If an integer $u \in \mathbb{Z}$ is a solution to the equation

$$P(x) = 0$$

then u is a divisor of the free term a_0, i.e. $u \mid a_0$.

Corollary 5.3. (The rational root theorem for monic polynomials) *Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, having the form*

$$P(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If a rational number $r \in \mathbb{Q}$ is a solution to the equation

$$P(x) = 0$$

then r is an integer and it is a divisor of the free term a_0 ($r \mid a_0$).

Example. Solve the following equation in real values of x:

$$x^4 - x^3 - 6x^2 + 14x - 12 = 0 \tag{5.1}$$

Solution. We try to find rational solutions of this equation using Corollary 5.3.

If equation (5.1) has a rational root x_1 then it is also integer, and it divides -12, i.e. we have

$$x_1 \mid (-12) \quad \Longrightarrow \quad x_1 \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}.$$

Now we try which of these is a solution of (5.1), if any. We do this using Horner's scheme:

	1	-1	-6	14	-12
1	1	0	-6	8	-4
-2	1	-3	0	14	-40
2	1	1	-4	6	0

So after trying 1 and -2 which are not solutions (the last value in the corresponding Horner's scheme is non-zero), we find that $x_1 = 2$ is a solution of (5.1), and the polynomial $x^4 - x^3 - 6x^2 + 14x - 12$ can be factorized as $(x-2)(x^3 + x^2 - 4x + 6)$, which means the (5.1) takes the form

$$(x - 2)(x^3 + x^2 - 4x + 6) = 0.$$

Since a product may be zero only if one of its factors is zero, thus we either have $x - 2 = 0$, which gives the expected solution

$$x_1 = 2,$$

or we have

$$x^3 + x^2 - 4x + 6 = 0. \tag{5.2}$$

This is a completely similar equation to the original equation (5.1), just its degree is smaller. So we try the same procedure again.

If equation (5.2) has a rational root x_2 then it is also integer, and it divides 6, i.e. we have

$$x_2 \mid 6 \quad \Longrightarrow \quad x_2 \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6\}.$$

We try which of these is a solution of (5.1), if any. We do this again using Horner's scheme:

	1	1	-4	6
2	1	3	2	10
-3	1	-2	2	0

Clearly, it is nonsense to try again $x = -2$ or $x = 1$, since if it was no root of the original polynomial, then it cannot be a root of one of its factors. On the other hand, we should try $x = 2$ again, since it could be a double root of the original polynomial. However, in this exercise this is not the case, and $x = 2$ is not a solution of (5.2). On the other hand we find that $x = -3$ is a solution of (5.2), and thus also of (5.1). More precisely, (5.2) may be written as

$$(x + 3)(x^2 - 2x + 2) = 0,$$

which either gives the solution

$$x_2 = -3,$$

or it leads to the quadratic equation

$$x^2 - 2x + 2 = 0,$$

which has discriminant $\Delta = -4$, so it has no real solutions. Thus the set of solutions of the (5.1) is

$$S = \{-3, 2\}.$$

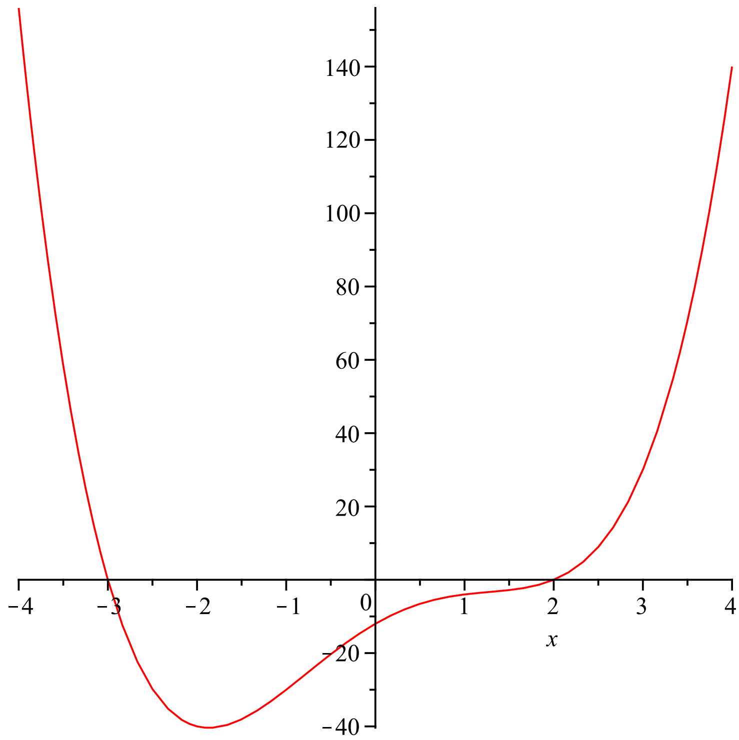

Figure 5.3: Graph of the polynomial $x^4 - x^3 - 6x^2 + 14x - 12$

The strategy of solving polynomial equations having rational solutions

1). Using Theorem 5.1 (or one of its Corollaries) we determine which are the possible rational solutions of the equation.

2). Then using Horner's scheme we try to decide if any of the above determined rational numbers is a solution.

3). If we find a solution we divide by the the corresponding linear factor, and we reduce the original equation to a similar equation of smaller degree.

4). We repeat the procedure to this equation.

5). We repeat the above steps until (hopefully) we get an equation which we can solve directly.

Exercise 5.1. Solve the following polynomial equations in real values of x:

a) $x^3 - 7x - 6 = 0$

b) $x^4 - 4x^3 - x^2 + 16x - 12 = 0$

c) $x^5 - 17x^3 + 12x^2 + 52x - 48 = 0$

d) $x^4 + 2x^3 - 15x^2 + 4x + 20 = 0$

e) $x^5 + 8x^4 + 3x^3 - 112x^2 - 308x - 240 = 0$

f) $x^6 + 3x^5 - 16x^4 - 25x^3 + 33x^2 + 52x + 60 = 0$

g) $x^6 - 28x^4 - 26x^3 + 147x^2 + 266x + 120 = 0$

h) $x^6 - 28x^4 - 26x^3 + 147x^2 + 266x + 120 = 0$

i) $x^7 + 2x^6 - 24x^5 - 66x^4 + 11x^3 + 144x^2 + 332x + 240 = 0$

j) $x^6 - 21x^4 + 31x^3 + 38x^2 - 79x + 30 = 0$

k) $x^6 + 16x^5 + 86x^4 + 205x^3 + 254x^2 + 184x + 64 = 0$

l) $x^6 - 2x^5 - 44x^4 - 55x^3 + 154x^2 + 232x + 64 = 0$

m) $x^6 - 2x^5 - 26x^4 + 40x^3 + 121x^2 - 254x + 120 = 0$

n) $x^7 + 3x^6 - 16x^5 - 35x^4 + 75x^3 + 56x^2 - 108x + 144 = 0$

o) $x^8 + 4x^7 - 18x^6 - 88x^5 - 167x^4 - 316x^3 - 216x^2 + 400x + 400 = 0$

p) $x^6 - 6x^5 + 6x^4 + 18x^3 - 31x^2 + 24x - 36 = 0$

q) $x^7 + 7x^6 + 9x^5 - 33x^4 - 82x^3 + 2x^2 + 120x + 72 = 0$

r) $x^6 - 11x^5 + 10x^4 + 84x^3 - 93x^2 - 153x + 162 = 0$

s) $x^6 + 29x^5 + 94x^4 - 156x^3 - 157x^2 - 185x - 250 = 0$

t) $x^8 + x^7 - 11x^6 - 4x^5 + 17x^4 - 61x^3 + 21x^2 + 144x - 108 = 0$

u) $x^6 + x^5 - 24x^4 - 25x^3 + 119x^2 + 144x + 144 = 0$

v) $x^6 + x^5 - 26x^4 - 29x^3 + 13x^2 + 100x + 300 = 0$

w) $x^4 + x^3 - 19x^2 - 25x - 150 = 0$

x) $x^4 + x^3 - 31x^2 - 36x - 180 = 0$

y) $x^5 - 6x^4 - 36x^3 + 204x^2 - 37x + 210 = 0$

z) $x^5 - 3x^4 - 45x^3 + 165x^2 - 46x + 168 = 0$

5.2.2 Biquadratic equations

Definition 5.4. Equation of the form

$$ax^4 + bx^2 + c = 0,$$

where $a \neq 0$ is called biquadratic.

Biquadratic equations is solved with method of introducing new variable: setting $x^2 = y$, we have a quadratic equation

$$ay^2 + by + c = 0.$$

5.2.3 Multi-quadratic equations

5.2.4 Reciprocal equations

Definition 5.5. A reciprocal equation of order n given by $f(x) = 0$ is one for which

$$f(x) = \pm x^n f\left(\frac{1}{x}\right).$$

It is clear that equation

$$f(x) = \sum_{k=0}^{n} a_k x^k = 0,$$

where $a_n \neq 0$, is a reciprocal equation if and only if

$$a_r = a_{n-r} \quad \text{or} \quad a_r = -a_{n-r}.$$

Such equations may be reduced to equations of lower degree by the substitutions

$$y = x + \frac{1}{x} \quad \text{or} \quad x - \frac{1}{x}.$$

Example. Solve the equation

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$$

Since $x = 0$ is not a solution, we can divide by x^2 and we get

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0.$$

Using the substitution $x + \frac{1}{x}$, we have the quadratic equation

$$2y^2 - 5y + 2.$$

The solutions are

$$y_1 = 2, y_2 = \frac{1}{2}.$$

In the first case we get for x

$$x + \frac{1}{x} = 2,$$

and so $x = 1$. If $y = \frac{1}{2}$ the corresponding quadratic equation is

$$x + \frac{1}{x} = \frac{1}{2},$$

however, it does not give real solutions for x.

5.3 Irrational equations

Definition 5.6. An equation with unknowns under the radical is called irrational equation.

There are two methods that are used to solve irrational equation:

1) Raising both sides of equation to a same power.

2) Introducing new variable.

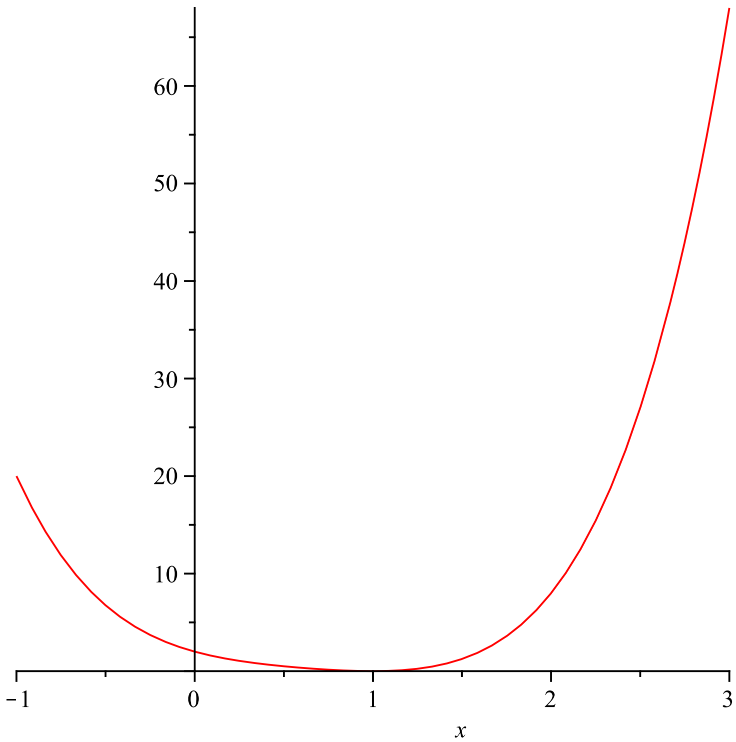

Figure 5.4: Graph of the function $2x^4 - 4x^3 + 6x^2 - 5x + 2$

Example. Solve the equation

$$\sqrt{x-1}+\sqrt{2x+6}=6.$$

Solution. We will rewrite the original equation a bit:

$$\sqrt{2x+6}=6-\sqrt{x-1}.$$

Square both sides we have

$$2x+6=36-12\sqrt{x-1}+x-1.$$

From this we get that

$$\sqrt{12x-1}=29-x.$$

Again square both sides we obtain

$$144(x-1)=(29-x)^2,$$

that is

$$x^2-202x+895=0.$$

one can see that this equation has two roots:

$$x_1=5, x_2=197.$$

After squaring, there could appear extraneous roots, so we need to check all roots. When $x=5$ we get that

$$\sqrt{5-1}+\sqrt{2\cdot 5+6}=6,$$

i.e. $x=5$ is root of our initial equation. When $x=197$ we have that

$$\sqrt{197-1}+\sqrt{2\cdot 197+6}\neq 6,$$

i.e. $x=197$ is an extraneous root. Therefore, initial equation has only one solution: $x=5$.

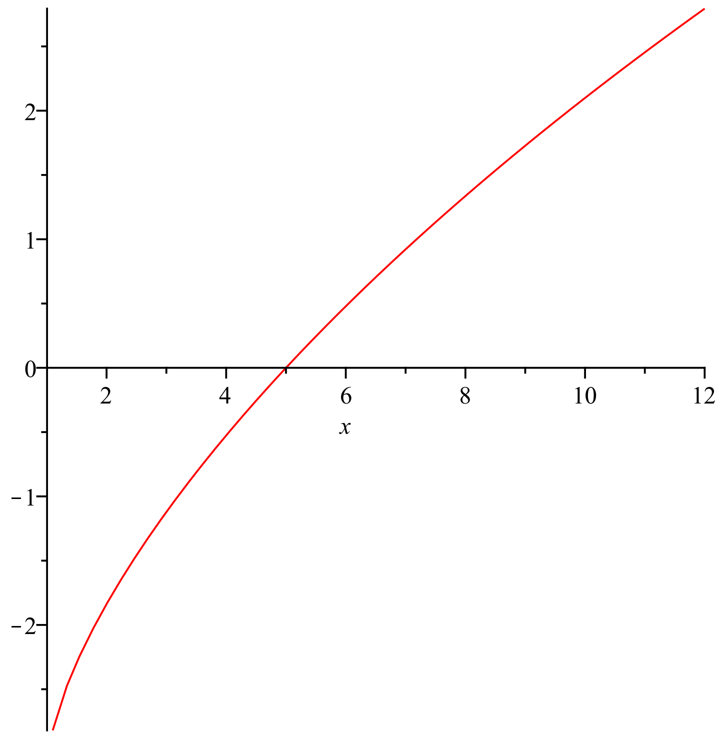

Figure 5.5: Graph of the function $\sqrt{x-1}+\sqrt{2x+6}-6$

Example. Solve the equation

$$x^2 + 3 - \sqrt{2x^2 - 3x + 2} = \frac{3}{2}(x + 4).$$

Solution. If we repeat the approach above, then we will face some difficult and tedious calculations. However, note that this equation can be transformed into quadratic. To do this multiply both sides of equation by 2 and we have

$$2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 12$$

or

$$2x^2 - 3x + 2 - 2\sqrt{2x^2 - 3x + 2} - 8 = 0.$$

Now, we introduce a new variable, let $y = 2x^2 - 3x + 2$, then equation can be rewritten as

$$y^2 - 2y - 8 = 0.$$

This equation has two solutions:

$$y_1 = 4, y_2 = -2.$$

Thus, we obtained set of equations:

$$\sqrt{2x^2 - 3x + 2} = 4, \sqrt{2x^2 - 3x + 2} = -2.$$

First equation possesses two possible solutions

$$x_1 = \frac{7}{2}, x_2 = -2.$$

Second equation does not have solutions, because square root cannot be negative.

We have to check our possible solutions. If $x = \frac{7}{2}$ then we have

$$\left(\frac{7}{2}\right)^2 + 3 - \sqrt{2 \cdot \left(\frac{7}{2}\right)^2 - 3 \cdot \frac{7}{2} + 2} = \frac{3}{2}\left(\frac{7}{2} + 4\right),$$

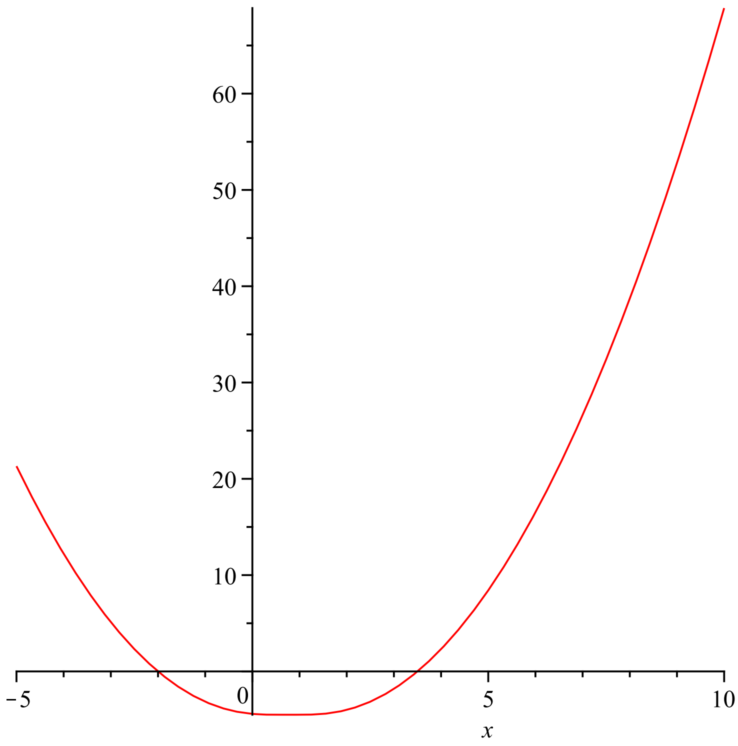

Figure 5.6: Graph of the function $x^2 - 3 - \sqrt{2x^2 - 3x + 2} - \frac{3}{2}(x+4)$

i. e. $x = \frac{7}{2}$ is a solution of the original equation.

If $x = -2$, the we get

$$((-2))^2 + 3 - \sqrt{2 \cdot (-2)^2 - 3 \cdot (-2) + 2} = \frac{3}{2}\left((-2) + 4\right),$$

so $x = -2$ is a solution of our original equation.

Example. Solve the equation

$$x^2 + x + \sqrt{x^2 + x + 7} = 5.$$

Solution: We will introduce a new variable, let $y = \sqrt{x^2 + x + 7}$ (The expression under square root is positive for every x). Rewriting our equation we have

$$y^2 - 7 + y = 5,$$

that is

$$y^2 + y - 12 = 0$$

which gives $y_1 = -4, y_2 = 3$. The negative value is impossible, the positive value gives

$$x^2 + x - 2 = 0$$

and we get

$$x_1 = 1, x_2 = -2.$$

Now, we need to check these candidates for the solution. If $x = 1$, then we have

$$1^2 + 1 + \sqrt{1^2 + 1 + 7} = 5,$$

this is a solution of our initial equation. If $x = -2$, then we obtain

$$(-2)^2 - 2 + \sqrt{(-2)^2 - 2 + 7} = 5,$$

so $x = 2$ is also a root of the original equation.

5.4 Exponential equations

Definition 5.7. An equation is called exponential equation if the unknown (or unknowns) appears in the exponents of algebraic expressions.

The following theorem is our basic tool for solving exponential equation.

Theorem 5.8. *Let b be a positive real number with $b \neq 1$. Then*

$$b^x = b^y \qquad \textit{implies} \qquad x = y$$

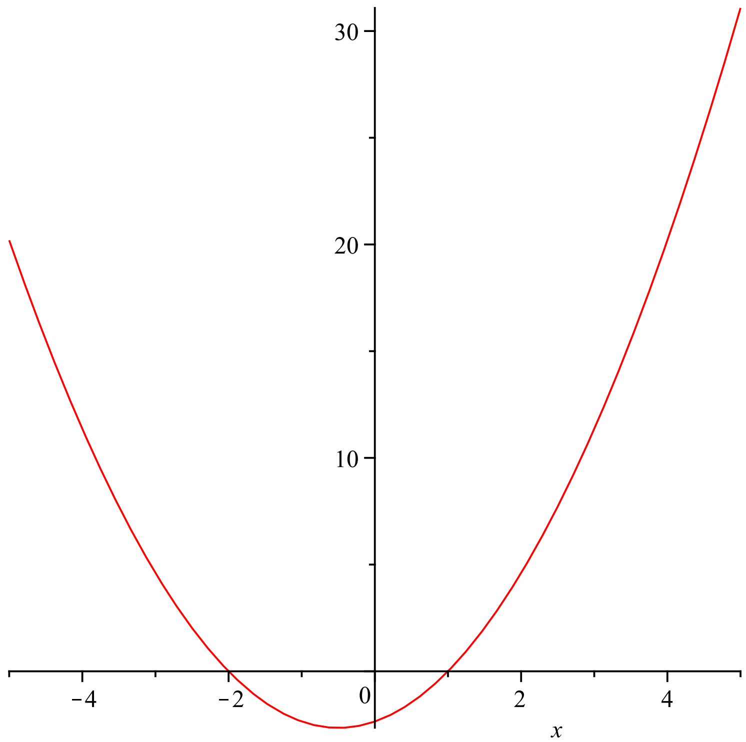

Figure 5.7: The graph of $x^2 + x + \sqrt{x^2 + x + 7} - 5$

The strategy of solving exponential equations

1). We try to transform our equation such that in the exponentiations all bases are the same.

2). Then we try to reduce the equation to several basic exponential equations.

3). We use Theorem 5.8 to get rid of the bases.

4). We solve the resulting equations.

5). We check the solutions by inserting them into the original equation.

In the sequel we shall solve several types of exponential equations.

Example. sok sok feladat

5.5 Logarithmic equations

5.5.1 Logarithms

When rising a number to a power our goal is to compute the result of an exponentiation. We also may reverse the question: which exponent to use for a base, to get a given result? This question motivates the definition of the logarithm.

Definition 5.9. Let b be a positive real number with $b \neq 1$, and let c be a positive number. The number a for which $b^a = c$ is called the **logarithm of** c **to the base** b, and it is denoted by $\log_b c$.

Remark. It is always useful to think of a logarithm as of an exponent, namely, the logarithm $\log_b c$ is the exponent which has to be put on the base b to get the result c.

Theorem 5.10. (Properties of logarithms) *Let b be a positive real number with $b \neq 1$. Further, let x, y be positive real numbers and z a real number. The we have the following properties of logarithms:*

$$1) \ \log_b y = z \iff b^z = y$$

$$2) \ \log_b(xy) = \log_b(x) + \log_b(y)$$

$$3) \ \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$4) \ \log_b\left(\frac{1}{y}\right) = -\log_b y$$

$$5) \ \log_b(x^z) = z \log_b x$$

$$6) \ \log_b 1 = 0 \qquad \log_b b = 1.$$

Remark. The statements of the above theorem may be formulated in words as follows:

1). The logarithm of y to the base b is the exponent which has to be placed on the base b to get the result y. (This is in fact the definition of the logarithm.)

2). The logarithm of a product is the sum of the logarithm of the factors.

3). The logarithm of a quotient is the difference between the logarithm of the numerator and the logarithm of the denominator.

4). The logarithm of the reciprocal of a positive real number is the negative of the logarithm of the number.

5). The logarithm of a power is the exponent times the logarithm of the base of the power.

6). The logarithm of 1 is 0, and the logarithm of the base itself is 1.

Remark. We mention that **there is no formula for** $\log_b(x+y)$ **and** $\log_b(x-y)$.

Theorem 5.11. (Changing the base of a logarithm) *Let a, b, c be positive real numbers with $a \neq 1$, $b \neq 1$ and $c \neq 1$. Further let x be a positive real number. Then we have*

$$1)\ \log_b x = \frac{\log_c x}{\log_c b}$$

$$2)\ \log_b a = \frac{1}{\log_a b}$$

Example. feladatok

Exercise 5.2. feladatok

5.5.2 Logarithmic equations

Definition 5.12. An equation is called **logarithmic equation** if the unknown (or unknowns) or algebraic expressions containing the unknown(s) appear in logarithm(s).

The following theorem is our basic tool for solving logarithmic equations.

Theorem 5.13. *Let b be a positive real number with $b \neq 1$. Then*

$$\log_b x = \log_b y \qquad \textit{implies} \qquad x = y,$$

and

$$\log_b x = a \qquad \textit{implies} \qquad x = b^a.$$

The strategy of solving logarithmic equations

1). We try to transform our equation such that the bases of all logarithms are the same.

2). Then we try to reduce the equation to several basic logarithmic equations.

3). We use Theorem 5.13 to get rid of the logarithms.

4). We solve the resulting equations.

5). We check the solutions by inserting them into the original equation.

In the sequel we shall solve several types of logarithmic equations.

Example. sok sok feladat

Chapter 6

Systems of equations

Chapter 7

Complex numbers

Definition 7.1. A complex number is an orderd pair (a, b) of real numbers, that is the set of complex numbers $\mathbb{C}$ is the Descartes product $\mathbb{R} \times \mathbb{R}$,

$$\mathbb{C} = \mathbb{R} \times \mathbb{R} = \{(a, b) : a, b \in \mathbb{R}\}.$$

Let $x = (a, b)$ and $y = (c, d)$ be two complex numbers. We write $x = y$ if $a = c$ and $b = d$ and we define the sum and product of two complex numbers as

$$x + y = (a + c, b + d),$$

$$xy = (ac - bd, ad + bc),$$

respectively.

Theorem 7.2. *With these definitions of addition and multiplication the set of complex number is a field with* $(0, 0)$ *and* $(1, 0)$ *in the role of* 0 *and* 1 *(cf. Section 1.2)*

Theorem 7.3. *For any real numbers* a *and* b *we have*

$$(a, 0) + (b, 0) = (a + b, 0),$$

and

$$(a, 0)(b, 0) = (ab, 0).$$

The previous theorem shows that the complex numbers of the form $(a, 0)$ have the same arithmetic structure as the corresponding real numbers a. We will therefore identify the complex number $(a, 0)$ with a. This identification dives us the field of real numbers as a subfield of the field of complex numbers.

In the sequel we define the mysterious square root of -1.

Definition 7.4.

$$i = (0, 1).$$

Indeed, one can check that

$$i^2 = i \cdot i = (0, 1)(0, 1) = (-1, 0) = -1.$$

Theorem 7.5. *If a and b are real, then*

$$(a, b) = a + bi.$$

Now we introduce the conjugate of a complex number.

Definition 7.6. If a, b are real and $z = a+bi$, then the complex number $\overline{z} = a-bi$ is called the conjugate of z. The numbers a and b are the real part and the imaginary part of z, respectively.

We will shortly write

$$a = \text{Re}(z), b = \text{Im}(z).$$

The most fundamental properties of the complex conjugate are

Theorem 7.7. *If z and w are complex, then*

(a) $\overline{z+w} = \overline{z} + \overline{w}$,

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$,

(c) $z + \overline{z} = 2\mathrm{Re}(z), z - \overline{z} = 2i\mathrm{Im}(z)$,

(d) $z\overline{z}$ *is nonnegative real number and* $z\overline{z}$ *is* 0 *if and only if* $z = 0$.

Using the last statement we can introduce the absolute value of a complex number. One can see that this definition is analogue to the usual absolute value for real numbers.

Definition 7.8. If z is a complex number, its absolute value $|z|$ is the nonnegative square root of $z\overline{z}$, that is

$$|z| = \sqrt{z\overline{z}}.$$

Note that x is a real number if and only if $x = \overline{x}$. Indeed, the equation

$$x + iy = x - iy$$

implies $y = 0$.

In the next theorem we summarize the most important properties of the absolute value.

Theorem 7.9. *Le z and w be complex numbers. Then we have*

(a) $z > 0$ *unless* $z = 0$, *in this case* $|0| = 0$,

(b) $|\overline{z}| = |z|$,

(c) $|zw| = |z||w|$,

(d) $|\mathrm{Re}(z)| \leq |z|$,

(e) $|z + w| \leq |z| + |w|$.

Chapter 8

Exercises for the interested reader

8.1 Identities of algebraic expressions

1). Let $a, b, c \in \mathbb{R}$ be real numbers. Prove that

(1) $(a+b+c)^3 - (a^3+b^3+c^3) = 3(b+c)(c+a)(a+b)$,

(2) $a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$.

2). Let $a, b, c \in \mathbb{R}$ be pairwise distinct non-zero real numbers. Prove that

(1) $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)} = 0$

(2) $\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)} = 1$

(3) $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} = a+b+c$

(4) $\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)} = a^2+b^2+c^2+ab+bc+ac$

(5) $\frac{a^{-1}}{(a-b)(a-c)} + \frac{b^{-1}}{(b-c)(b-a)} + \frac{c^{-1}}{(c-a)(c-b)} = \frac{1}{abc}$

(6) $\frac{a^{-2}}{(a-b)(a-c)} + \frac{b^{-2}}{(b-c)(b-a)} + \frac{c^{-2}}{(c-a)(c-b)} = \frac{ab+bc+ac}{a^2b^2c^2}$

3). Let $a, b, c, d \in \mathbb{R}$ be pairwise distinct non-zero real numbers. Prove that

(1) $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} = 0$

(2) $\frac{1}{(a-b)(a-c)(a-d)} + \frac{1}{(b-a)(b-c)(b-d)} + \frac{1}{(c-a)(c-b)(c-d)} + \frac{1}{(d-a)(d-b)(d-c)} = 0$

(3) Generalize the above statements

4). Let $a, b, c, d \in \mathbb{R}$ be real numbers, such that the below fractions have non-zero denominators. Prove that

(1) $\frac{b}{a(a+b)} + \frac{c}{(a+b)(a+b+c)} = \frac{b+c}{a(a+b+c)}$

(2) $\frac{b}{a(a+b)} + \frac{c}{(a+b)(a+b+c)} + \frac{d}{(a+b+c)(a+b+c+d)} = \frac{b+c+d}{a(a+b+c+d)}$

(3) Generalize the above statements

5). Let $a, b, c, d \in \mathbb{R}$ be real numbers, such that the expressions in the following exercises have sense.

(1) If $a + b = 1$ then compute the value of the expression

$$a^3 + b^3 + 3(a^3b + ab^3) + 6(a^3b^2 + a^2b^3)$$

(2) If $a, d \in \mathbb{Z}$ then prove that the sum

$$a^2 + 2(a + d)^2 + 3(a + 2d)^2 + 4(a + 3d)^2$$

can be written as the sum of two perfect squares.

(3) Prove that if $a + b + c = 0$ then

$$a^3 + a^2c + b^2c - abc + b^3 = 0.$$

(4) Prove that if the difference of two integers is 2 then the difference of their cubes can be written as the sum of three perfect squares.

6). Compute the value of the expression

$$\left(\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}\right)\cdot\left(\frac{c}{a-b}+\frac{a}{b-c}+\frac{b}{c-a}\right)$$

provided that

(1) $a+b+c=0$,

(2) $|c|=|a-b|$.

7). Let $a,b,c\in\mathbb{R}$

8). Let $a,b,c\in\mathbb{R}$ be real numbers. Prove that if $(a+b+c)^3=(a^3+b^3+c^3)$ then for every $n\in\mathbb{N}$ we have

$$(a+b+c)^{2n+1}=a^{2n+1}+b^{2n+1}+c^{2n+1}.$$

9). Let $a,b,c\in\mathbb{R}$ be real numbers, such that the expressions below have sense. Prove that if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$$

then for every $n\in\mathbb{N}$ we have

$$\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$$

10). Let $a,b,c\in\mathbb{R}$ be real numbers. Prove that if $a^3+b^3+c^3=3abc$ then either $a+b+c=0$ or $a=b=c$.

Ide meg a Stamate Stolanbol

8.2 Inequalities of algebraic expressions

1). Let $a, b, c \in \mathbb{R}$ be positive real numbers. Prove that

(1) $(a+b)(a+c)(c+a) \geq 8abc$,

(2) $(a^2+b^2)c + (b^2+c^2)a + (c^2+a^2)b \geq 6abc$,

(3) $2(a^3+b^3+c^3) \geq (a+b)ab + (b+c)bc + (c+a)ca$.

2). Let $a, b, c \in \mathbb{R}$ be positive real numbers with $a+b+c=1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

Under what conditions do we have equality above?

3). Let $a, b \in \mathbb{R}$ be positive real numbers with $a+b=2$. Prove that

$$a^4 + b^4 \geq 2.$$

4). Let $a, b \in \mathbb{R}$ be positive real numbers, and $m, n \in \mathbb{N}$ be natural numbers of the same type of parity. Prove that

$$(1) \qquad \frac{a^m+b^m}{2} \cdot \frac{a^n+b^n}{2} \leq \frac{a^{m+n}+b^{m+n}}{2}$$

$$(2) \qquad \frac{a+b}{2} \cdot \frac{a^2+b^2}{2} \cdot \frac{a^3+b^3}{2} \leq \frac{a^6+b^6}{2}$$

5). Let $a, b, c \in \mathbb{R}$ be positive real numbers. Prove that

$$a+b+c \leq \frac{a^4+b^4+c^4}{abc}$$

6). Let $a, b, c, d \in \mathbb{R}$ be positive real numbers. Prove that

$$\sqrt{(a+c)(b+d)} \leq \sqrt{ab} + \sqrt{cd}.$$

7). Let $a, b, c \in \mathbb{R}$ be positive real numbers. Prove that

$$\frac{ab}{a+c} + \frac{bc}{b+c} + \frac{ca}{c+a} = \frac{a+b+c}{2}.$$

Under what conditions do we have equality above?

8). Prove that

(1) $ab + ac + bc \leq a^2 + b^2 + c^2$ for any $a, b, c \in \mathbb{R}$

(2) $ab + ac + ad + bc + bd + cd \leq \frac{3}{2}(a^2 + b^2 + c^2 + d^2)$ for any $a, b, c, d \in \mathbb{R}$,

(3) $\sum\limits_{i=1}^{n} \sum\limits_{j=i+1}^{n} a_i a_j \leq \frac{n-1}{2} \sum\limits_{i=1}^{n} a_i^2$.

9). Let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$ be rational numbers with $a_i, b_i \in \mathbb{Z}$, $b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\min_{1 \leq i \leq n} \frac{a_i}{b_i} \leq \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \max_{1 \leq i \leq n} \frac{a_i}{b_i}$$

10). Ide meg kozepertekes feladokat, sokat.

Chapter 9

Results of the Exercises

9.1 Chapter 1

Results of Exercise 1.1: The listed elements belong to the corresponding intervals, and those elements form the list $-7, -5.3, -5, -4.99, -\pi, -1, 0, 1, \sqrt{3}$, 3.99, $4, 4.02, 5$ which are not listed, do not belong to the corresponding interval.

a) $-7, -5.3 \in \,]-\infty, -5[$

b) $-7, -5.3, -5 \in \,]-\infty, -5]$

c) $-4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99 \in \,]-5, 4[$

d) $-5, -4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99 \in [-5, 4[$

e) $-4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99, 4 \in \,]-5, 4]$

f) $-5, -4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99, 4 \in [-5, 4]$

g) $-5, -4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99, 4, 4.02, 5 \in [-5, \infty[$

h) $-4.99, -\pi, -1, 0, 1, \sqrt{3}, 3.99, 4, 4.02, 5 \in \,]-5, \infty[$

f) $\sqrt{3}, 3.99, 4, 4.02 \in \left[\sqrt{3}, 5\right[$

We leave for the students to draw the graph representing the above intervals on a real number line.

Results of Exercise 1.2:

a) $x \in]2, 7[$	b) $x \in [3, \infty[$	c) $x \in]-\infty, 3]$
d) $x \in [5, 8[$	e) $x \in]5, 11]$	f) $x \in]-\infty, 7[$
g) $x \in]-1, \infty[$	h) $x \in]-1, \infty[$	f) $x \in [1, 3]$

Results of Exercise 1.3

a) 2^{13}	b) 2^2	c) 2^6
d) 2^{12}	e) 2^4	f) 1
g) 3^2	h) 3^{32}	i) 3^{10}
j) $2^3 \cdot 3^2 \cdot 5^0 \cdot 7^2$	k) 2^4	l) $9a^4b^8c^6$
m) $-12a^3b^5c^9$	n) $9a^4b$	o) $\dfrac{9a}{8b^2c} = \dfrac{9}{8}ab^{-2}c^{-1}$
p) $-\dfrac{1}{3}a^{14}b^4c$	q) $10a^3b^2d^{-3}$	r) $a^5b^2c^2$
s) $-\dfrac{1}{8}x^{-8}y^{17}z^{13}$	t) $6x^{11}y^{12}z^7$	u) a^8b^2

Results of Exercise 1.4

a) $\sqrt[6]{2}$	b) $\sqrt[10]{3}$	c) $\sqrt[3]{a}$
d) $\sqrt[12]{a^2b}$	e) $\sqrt[8]{a}$	f) $\sqrt[24]{a^{23}}$
g) $\sqrt[8]{\dfrac{x^3}{y^3}}$	h) $\sqrt[12]{a^7b^9}$	i) $\sqrt[6]{\dfrac{a^2}{b}}$
j) $\sqrt[30]{x^{13}y^7}$	k) $\sqrt[24]{\dfrac{x^9}{y^7}}$	l) $\sqrt[6]{x}$
m) $\sqrt[24]{a^{17}}$	n) a^4	o) $\sqrt[4]{a}$
p) $\sqrt[4]{a^5}$	q) $\sqrt[75]{a^{57}}$	r) $\sqrt[24]{a^{35}}$

Results of Exercise 1.5:

a) ab	b) $a^{-1}b$	c) $a^2b^{-\frac{1}{2}}$
d) a^2b^{-3}	e) ab	f) $a^{-2}b^{-\frac{4}{5}}$
g) $a^{\frac{11}{3}}b^{\frac{11}{2}}$	h) $a^{-\frac{4}{3}}b$	i) $a^{\frac{80}{21}}b^{\frac{8}{15}}$
j) 1	k) $a^{\frac{3}{5}}b^{\frac{15}{4}}$	l) 1

9.2 Chapter 2

Results of Exercise 2.1:

a) $2x^2 - 2xy^3 + 2xy + 3x + y^2 + y$

b) $3x^3 - 4x^2y + 4xy^2 + 5y^3$

c) $2x^5 - 7x^4y + 8x^3y^2 - 5x^2y^3 - 12xy^4 + 5y^5$

d) $-x^2 + 4xy^3 - 13xy - 6x + y^2 - 2y$

e) $x^4 - 2x^3y^3 + 2x^3y + 3x^3 + 6x^2y^4 - 14x^2y^2 - 8x^2y - 2xy^5 + 5xy^3 + y^3$

f) $2x^3y^3 - 8x^3y - 3x^3 - 6x^2y^4 + 25x^2y^2 + 8x^2y + 2xy^5 - 11xy^3 + y^4 - y^3$

Results of Exercise 2.2:

a) $x^3 - x^2 + 7$

b) $2x^3 - 5x^2 + 5x + 5$

c) $x^5 - 5x^4 + 11x^3 - 8x^2 - 9x + 10$

d) $3x^2 - 3x + 6$

e) $x^4 - 3x^3 + 4x^2 - 6x + 4$

f) $x^5 - x^4 + 2x^3 + 5x^2 + 14$

g) $-2x^5 + 3x^4 - 11x^3 - 9x^2 - 14x - 30$

h) $x^3 + 9$

i) $x^5 - 4x^4 + 8x^3 - 4x^2 - 15x + 14$

j) $x^7 - 5x^6 + 13x^5 - 18x^4 + 13x^3 - 6x^2 - 18x + 20$

Results of Exercise 2.3:

a) $4xy^3$	b) $2y^4z$	c) $\frac{1}{2}x^2y^2z$
d) $\frac{3}{5}ab^4$	e) $2a^2b^4c^2$	f) $\frac{4}{5}a^2c^5d^2$

Results of Exercise 2.4 Denoting the quotient by $q(x)$ and the remainder by

$r(x)$ the results of the divisions are:

a) $q(x) = x + 3,$ $r(x) := -2x - 1$

b) $q(x) = x^3 - 2x^2 + x + 5,$ $r(x) = -3$

c) $q(x) = x^3 - 2x - 4,$ $r(x) = -12x + 10$

d) $q(x) = x^3 - x - 1,$ $r(x) = -5x + 3$

e) $q(x) = x^3 + 2x + 1,$ $r(x) = 0$

f) $q(x) = x^4 + 2x^3 - 8x - 16,$ $r(x) = 0$

g) $q(x) = x^3 + x^2 - x - 7,$ $r(x) = -20x + 30$

h) $q(x) = x^3 + 7x^2 + 28x + 126,$ $r(x) = 574x - 250$

i $q(x) = x^3 + 2x^2 - 4,$ $r(x) = -19x + 14$

j) $q(x) = x + 1,$ $r(x) = -3x^3 + 5x^2 - 2x - 1$

k) $q(x) = x^5 + 4x^4 + 2x^3 - 3x^2 - x - 4,$ $r(x) = 6$

l) $q(x) = x^5 + 2x^4 - 4x^3 - x^2 + 3x - 6,$ $r(x) = 16$

m) $q(x) = x^5 + 5x^4 + 8x^3 + 11x^2 + 24x + 45,$ $r(x) = 100$

n) $q(x) = x^5 + x^4 - 4x^3 + 3x^2 - 4x + 5,$ $r(x) = 0$

o) $q(x) = x^5 - 2x^3 + x^2 - x,$ $r(x) = 10$

p) $q(x) = x^4 + 3x^3 - x^2 - 2x + 1,$ $r(x) = -5x + 11$

q) $q(x) = x^5 + x^4 - 3x^3 - 4x^2 - 4x - 1,$ $r(x) = -3$

r) $q(x) = x^5 - x^4 - 3x^3 + 2x^2 - 2x + 5,$ $r(x) = -7$

s) $q(x) = x^5 + 2x^4 - x^2 - 2x - 1,$ $r(x) = -4$

t) $q(x) = x^5 - 2x^4 - x^2 + 2x - 1,$ $r(x) = 0$

u) $q(x) = x^5 + 3x^4 + 5x^3 + 14x^2 + 42x + 129,$ $r(x) = 385$

v) $q(x) = x^5 - 3x^4 + 5x^3 - 16x^2 + 48x - 141,$ $r(x) = 421$

w) $q(x) = x^4 + x^3 - 2x^2 - 2x - 4,$ $r(x) = -3x - 6$

x) $q(x) = x^4 - 3x^2 - x - 3,$ $r(x) = 2x - 5$

y) $q(x) = x^4 - x^3 - 2x^2 - 2,$ $r(x) = 5x - 4$

z) $q(x) = x^4 - 2x^3 + x^2 - 5x + 11,$ $r(x) = -24x + 9$

Results of Exercise 2.5 Denoting the quotient by $q(x)$ and the remainder by

$r(x)$ the results of the divisions are:

a) $q(x) = x^4 - 3x^3 + 4x^2 - 7x + 9, \quad r(x) = -14$

b) $q(x) = x^4 - 3x^3 - 3x^2 - 8x - 14, \quad r(x) = -31$

c) $q(x) = x^4 - 7x^3 + 17x^2 - 36x + 74, \quad r(x) = -151$

d) $q(x) = x^6 - 4x^5 - 2x^4 - 6x^3 - 7x^2 - 7x - 4, \quad r(x) = -6$

e) $q(x) = x^6 - 6x^5 + 8x^4 - 12x^3 + 11x^2 - 11x + 14, \quad r(x) = -16$

f) $q(x) = x^6 - 3x^5 - 4x^4 - 12x^3 - 25x^2 - 50x - 97, \quad r(x) = -196$

g) $q(x) = x^6 - 7x^5 + 16x^4 - 36x^3 + 71x^2 - 142x + 287, \quad r(x) = -576$

h) $q(x) = x^5 + x^4 + x^3 + x^2 + x + 1, \quad r(x) = -1$

i) $q(x) = x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32 \quad r(x) = 62$

j) $q(x) = x^5 - 2x^4 + 4x^3 - 8x^2 + 16x - 32, \quad r(x) = 62$

k) $q(x) = x^5 + 4x^4 + 2x^3 - 3x^2 - x - 4, \quad r(x) = 6$

l) $q(x) = x^5 + 2x^4 - 4x^3 - x^2 + 3x - 6, \quad r(x) = 16$

m) $q(x) = x^5 + 5x^4 + 8x^3 + 11x^2 + 24x + 45, \quad r(x) = 100$

n) $q(x) = x^5 + x^4 - 4x^3 + 3x^2 - 4x + 5, \quad r(x) = 0$

o) $q(x) = x^5 - 2 * x^3 + x^2 - x, \quad r(x) = 10$

p) $q(x) = x^6 - x^5 + x^2 - x, \quad r(x) = 0$

q) $q(x) = x^5 + x^4 - 3x^3 - 4x^2 - 4x - 1, \quad r(x) = -3$

r) $q(x) = x^5 - x^4 - 3x^3 + 2x^2 - 2x + 5, r(x) = -7$

s) $q(x) = x^5 + 2x^4 - x^2 - 2x - 1, \quad r(x) = -4$

t) $q(x) = x^5 - 2x^4 - x^2 + 2x - 1, \quad r(x) = 0$

u) $q(x) = x^5 + 3x^4 + 5x^3 + 14x^2 + 42x + 129, \quad r(x) = 385$

v) $q(x) = x^5 - 3x^4 + 5x^3 - 16x^2 + 48x - 141, \quad r(x) = 421$

w) $q(x) = x^6 + 2x^5 - 3x^3 + 4x^2 - 9x + 6, \quad r(x) = -2$

x) $q(x) = x^6 + 4x^5 + 6x^4 + 3x^3 + 4x^2 - x - 4, \quad r(x) = 0$

y) $q(x) = x^6 + x^5 - 3x^3 + 7x^2 - 19x + 35, \quad r(x) = -66$

z) $q(x) = x^6 + 2x^5 - 3x^3 + 4x^2 - 9x + 6, \quad r(x) = 0$

Results of Exercise 2.6 Denoting the quotient by $q(x)$ and the remainder (when it is non-zero) by $r(x)$ the results of the divisions are:

a) no $q(x) = x^5 + 6x^4 + x^3 - 29x^2 - 62x - 113, \quad r(x) = -196$

b) yes $q(x) = x^5 + 2x^4 - 15x^3 - x^2 - 2x + 15$

c) yes $q(x) = x^5 + 5x^4 - 6x^3 - 37x^2 - 41x - 30$

d) no $q(x) = x^5 + 3x^4 - 14x^3 - 17x^2 + 13x - 2, \quad r(x) = 32$

e) no $q(x) = x^5 + x^4 - 14x^3 + 11x^2 - 37x + 122, \quad r(x) = -336$

f) yes $q(x) = x^5 + 7x^4 + 10x^3 - x^2 - 7x - 10$

g) no $q(x) = x^5 - 11x^3 + 13x^2 - 56x + 235, \quad r(x) = -910$

h) yes $q(x) = x^5 - x^4 - 6x^3 - x^2 + x + 6$

i) yes $q(x) = x^5 + 3x^4 - 6x^3 - 25x^2 - 27x - 18$

j) no $q(x) = x^5 + 3x^4 - 3x^3 - 16x^2 - 33x - 57, \quad r(x) = -105x - 210$

k) yes $q(x) = x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2$

l) yes $q(x) = x^6 + 7x^5 + 22x^4 + 43x^3 + 49x^2 + 34x + 12$

m) yes $q(x) = x^6 + 5x^5 + 10x^4 + 11x^3 - 5x^2 - 10x - 12$

n) yes $q(x) = x^5 + 6x^4 + 16x^3 + 27x^2 + 22x + 12$

o) no $q(x) = x^6 + 8x^5 + 31x^4 + 83x^3 + 172x^2 + 329x + 636, \quad r(x) = 1260$

p) no $q(x) = x^6 + 4x^5 + 7x^4 + 7x^3 - 8x^2 + x - 24, \quad r(x) = 36$

q) no $q(x) = x^5 + 6x^4 + 19x^3 + 45x^2 + 82x + 165, \quad r(x) = 306x + 648$

r) no $q(x) = x^6 + 9x^5 + 42x^4 + 147x^3 + 447x^2 + 1326x + 3956, \quad r(x) = 11856$

s) yes $q(x) = x^6 + 3x^5 + 6x^4 + 3x^3 - 3x^2 - 6x - 4$

t) no $q(x) = x^5 + 6x^4 + 24x^3 + 75x^2 + 222x + 660, \quad r(x) = 1976x + 5928$

u) no $q(x) = x^7 - 11x^5 - x^4 + 12x^3 + 11x^2 + 48x - 12, \quad r(x) = 48$

v) no $q(x) = x^7 + 2x^6 - 9x^5 - 21x^4 - 10x^3 + 13x^2 + 72x + 108, \quad r(x) = 144$

w) no $q(x) = x^6 + x^5 - 10x^4 - 11x^3 + x^2 + 12x + 60, \quad r(x) = 48x + 96$

x) yes $q(x) = x^6 + x^5 - 7x^4 - 8x^3 - 17x^2 - 9x - 9$

y) yes $q(x) = x^6 + x^5 - 2x^4 - 3x^3 - 7x^2 - 4x - 4$

z) yes $q(x) = x^6 - 4x^5 + 3x^4 - 3x^3 + 8x^2 + x + 6$

Results of Exercise 2.7:

a) 144	b) 0	c) 0
d) 18564	e) 0	f) 0
g) 48	h) 36	i) 64
j) -80	k) 70	l) -110
m) 0	n) 0	o) 1120

Results of Exercise 2.8

a) $5(x^2y+3x-1)$	b) $8(a-b+2)$	c) $4x^2(2x-3y^2+z)$
d) $a^2b(a+b^2+1)$	e) $a(x+y)(1-b)$	f) $2m^2(m-2m^3n+)$
g) $x^2y(xy-5+x^2y)$	h)) $2a(b^2-2)(2a-b)$	i) $(x-1)(a-b-7)$
j) $(x+5)(x-5)$	k) $(x+3y)(x-3y)$	l $(x-2)(x+2)(x^2+4)$
m) $(2a-3b)^2$	n) $(a-2b)^3$	o) $(5a+2b)(25a^2-10ab+4b^2)$
p) $(9a+4b)(9a-4b)$	q) $(5x+6y)(5x-12y)$	r) $(82x-15y)(58x-45y)$
s) $(a+b)(c+d)$	t) $(a+b)(c-d)$	u) $(a^2+2)(a+2)$
v) $(a+b)^2(a-b)$	w) $(x-z)(x^2+2z^2)$	x) $(a+b)(a-b)(a^2+ab+b^2)$
y) $(x-1)(x-3)$	z) $(x-3)(x+2)$	ω) $(x+1)(x-1)(x+2)(x-2)$

Results of Exercise 2.9

a) $3x(1+3x^2y^3)^2$

b) $(x-y+z)(xy-10)(xy+10)$

c) $(3a+2)(3a-2)(a^2+5)$

d) $(a-1)(a+1)(b-1)(b+1)(a^2+1)(b^2+1)$

e) $7(4x+y)(x^2-xy+y^2)$

f) $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$

g) $2(x^2+xy+y^2)^2$

h) $(x+a+b)(x+a-b)(x-a+b)(x-a-b)$

i) $x(1+x)(1-x)(x-3)$

j) $(x-1)^2(x-4)(x^2+x+1)$

k) $(x+y)(x+y-z)$

l) $(ab-cd)(bc-ad)(ac-bd)$

m) $(x^2y^2+x^4-y^4)(x^2y^2-x^4+y^4)$

n) $(a^2+pb^2)(c^2+pd^2)$

o) $(c+b)(c-b)(a-c)$

p) $(x+2)(x+4)(x^2+5x+8)$

q) $a^2c^2(b+c)(b-c)(a-c)$

r) $(ab-cd+ac+bd)(ab-cd-ac-bd)$

s) $x(x^2+x+1)^2$

t) $(3x-1)(3x+1)(x^2+x+1)^2$

u) $(2x-3y)(4x^2-6xy+9y^2)$

v) $(cx+by)(ax+cy)(bx+ay)-(bx+cy)(cx+ay)(ax+by)$

v) $xy(x-y)(a-b)(a-c)(c-b)$

w) $(x-1)(x+2)(x^2+x+5)$

x) $(1-ab)(1-bc)(1-ca)$

y) $x(x+1)^2(x^2+1)$

z) $(a-b)(a-c)(b-c)(a+b+c)$

Results of Exercise 2.10

a) $\dfrac{5x}{y}$ **b)** $\dfrac{y^6z^3}{2x^2w^2}$

c) $\dfrac{-1}{2x^2}$ **d)** $\dfrac{1}{-4x}$

e) $\dfrac{1}{2(x-3)}$ **f)** $\dfrac{5(x-1)}{x+1}$

g) $\dfrac{3x^2+xy}{xy+3y^2}$ **h)** $\dfrac{1}{x}$

i) $\dfrac{5}{x+4}$ **j)** $\dfrac{7(x^2-y^2)}{9y^2}$

Results of Exercise 2.11

a) $\dfrac{3a}{5a^3b^7}$ **and** $\dfrac{5b^6}{5a^3b^7}$

b) $\dfrac{x+1}{a^3(x+1)^2}$ **and** $\dfrac{a}{a^3(x+1)^2}$

c) $\dfrac{(x+3)(3x+2)}{(2x-1)(3x+2)}$ **and** $\dfrac{(x-1)(2x-1)}{(2x-1)(3x+2)}$

d) $\dfrac{3a^3}{3a^3(x+1)}$, $\dfrac{3(x+1)}{3a^3(x+1)}$ **and** $\dfrac{a^2(x+1)}{3a^3(x+1)}$

Results of Exercise 2.12

a) ??

b) ??

c) $\dfrac{a+3}{a-3}$ **d)** $(a+b)(x+1)$

e) $\dfrac{xy(x+y)}{y-1}$

f) $\dfrac{x+1}{x+y}$

g) $\dfrac{1}{a^2}$

h) $a+b$

i) $\dfrac{1}{2ab}$

j) 5

k) $\dfrac{b}{b-a}$

l) $x-y$

m) $\dfrac{1}{ab}$

n) $x+1$

o) 1

p) $-x$

q) 0

Results of Exercise 2.13

a) $\dfrac{\sqrt{7}}{7}$ **b)** $\dfrac{\sqrt[3]{25}}{5}$ **c)** $\dfrac{\sqrt[k]{a^{k-1}}}{a}$

Results of Exercise 2.14

a) $\frac{\sqrt{5}+\sqrt{2}}{3}$

b) $-\frac{1+\sqrt{5}}{4}$

c

d) $sqrt5-\sqrt{2}$

e) $\sqrt{6}-1$

f

g) $\frac{\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}}{3}$

h) $\frac{\sqrt[3]{49}+\sqrt[3]{28}+\sqrt[3]{16}}{4}$

i

j) $\frac{\sqrt[4]{125}+\sqrt[4]{50}+\sqrt[4]{20}+\sqrt[4]{8}}{3}$

k) $\frac{\sqrt[5]{625}+\sqrt[5]{250}+\sqrt[5]{100}+\sqrt[5]{40}+\sqrt[5]{16}}{3}$

l

m) $\frac{(\sqrt[4]{5}-\sqrt[4]{2})(\sqrt{5}+\sqrt{2})}{3}$

n)

o

p)

q)

r

s)

t)

u

v)

w)

x

y)

z)

α

β)

γ)

δ

Results of Exercise 2.15

a) 2

b) 4

c) 4

d)

e) $\sqrt{2}$

f) 4

g) 6

h)

i)

j)

k)

l)

m)

n)

Results of Exercise 2.16

a)

b)

c)

d)

9.3 Chapter 3

Results of Exercise 3.1

a) $S = \{4\}$

b) $S = \{4\}$

c) $S = \{3\}$

d) $S = \{4\}$

e) $S = \mathbb{Q}$

f) $S = \emptyset$

g) $S = \{4\}$

h) $S = \emptyset$

i $S = \left\{\frac{17}{9}\right\}$

j) $S = \{-2\}$

k) $S = \left\{\frac{7}{8}\right\}$

l) $S = \{-2\}$

m) $S = \emptyset$

n) $S = \left\{\frac{1}{9}\right\}$

o) $S = \{3\}$

p) $S = \left\{-\frac{271}{10}\right\}$

q) $S = \{5\}$

r) $S = \{2\}$

s) $S = \left\{-3, \frac{2}{5}\right\}$

t) $S = \{4\}$

u) $S = \left\{-2, -3, \frac{1}{2}\right\}$

Results of Exercise 3.2

a) $S = \{-1, 1\}$

b) $S = \{-5, 5\}$

c) $S = \{-\sqrt{2}, \sqrt{2}\}$

d) $S = \emptyset$

e) $S = \{0, 3\}$

f) $S = \{-5, 0\}$

g) $S = \{1\}$

h) $S = \{1, 3\}$

i) $S = \{1, 4\}$

j) $S = \{3, 4\}$

k) $S = \{-3, 1\}$

l) $S = \{-6, 1\}$

m) $S = \{-1, 6\}$

n) $S = \{-3\}$

o) $S = \emptyset$

p) $S = \emptyset$

q) $S = \left\{\frac{1}{2}, 1\right\}$

r) $S = \left\{-\frac{1}{2}, \frac{2}{3}\right\}$

s) $S = \left\{-\frac{3}{5}, \frac{4}{3}\right\}$

t) $S = \left\{-\frac{5}{2}, -2\right\}$

u) $S = \left\{\frac{-1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right\}$

v) $S = \emptyset$

w) $S = \emptyset$

x) $S = \left\{\frac{1}{2}\right\}$

y) $S = \left\{-\frac{2}{3}\right\}$

z) $S = \left\{\frac{7}{2}\right\}$

Results of Exercise 3.3

a) $S = \left\{\frac{-3}{2}, 2\right\}$

b) $S = \left\{-\sqrt{5}, \sqrt{5}\right\}$

c) $S = \{-3, -4\}$

d) $S = \{-5, 3\}$

e) $S = \{8\}$

f) $S = \emptyset$

g) $S = \{3\}$

h) $S = \left\{\frac{6}{5}, \frac{12}{5}\right\}$

i $S = \{-1, 1\}$

j) $S = \emptyset$

9.4 Chapter 4

Results of Exercise 4.2:

a) $x \in\ ]-\infty, 1[\ \cup\ ]3, \infty[$

b) $x \in\ ]-\infty, -2] \cup [-1, \infty[$

c) $x \in\ ]-3, -2[$

d) $x \in [-3, 2]$

e) $x \in\ ]-\infty, -2] \cup [2, \infty[$

f) $x \in\ ]-3, 3[$

g) $x \in \mathbb{R}$

h) $x = 3$

i) $x \in \mathbb{R} \setminus \{-2\}$

j) $x \in \emptyset$

k) $x \in \emptyset$

l) $x \in \mathbb{R}$

m) $x \in \mathbb{R}$

n) $x \in \mathbb{R}$

o) $x \in \emptyset$

p) $x \in \mathbb{R}$

9.5 Chapter 5

Results of Exercise 5.1:

a) $x_1 = -2, x_2 = -1, x_3 = 3$

b) $x_1 = -2, x_2 = 1, x_3 = 2, x_4 = 3$

c) $x_1 = -4, x_2 = -2, x_3 = 1, x_4 = 2, x_5 = 3$

d) $x_1 = 2, x_2 = 2, x_3 = -1, x_4 = -5$

e) $x_1 = -5, x_2 = -3, x_3 = -2, x_4 = -2, x_5 = 4$

f) $x_1 = -5, x_2 = -2, x_3 = 2, x_4 = 3$

g) $x_1 = -4, x_2 = -2, x_3 = 3, x_4 = 5$

h) $x_1 = -3, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = -1 - \sqrt{3}, x_6 = -1 + \sqrt{3}$

i) $x_1 = -4, x_2 = -3, x_3 = -1, x_4 = 2, x_5 = 5$

j) $x_1 = -5, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = \dfrac{-1 + \sqrt{5}}{2}, x_6 = \dfrac{-1 - \sqrt{5}}{2}$

k) $x_1 = -1, x_2 = -2, x_3 = -4, x_4 = -8$

l) $x_1 = -1, x_2 = 2, x_3 = -4, x_4 = 8, x_5 = \dfrac{-3 - \sqrt{5}}{2}, x_6 = \dfrac{-3 + \sqrt{5}}{2}$

m) $x_1 = -4, x_2 = -3, x_3 = 1, x_4 = 1, x_5 = 2, x_6 = 5$

n) $x_1 = -4, x_2 = -3, x_3 = -2, x_4 = 2, x_5 = 3$

o) $x_1 = -5, x_2 = -2, x_3 = -2, x_4 = -1, x_5 = 1, x_6 = 5$

p) $x_1 = -2, x_2 = 2, x_3 = 3, x_4 = 3$

q) $x_1 = -3, x_2 = -3, x_3 = -2, x_4 = -1, x_5 = 2, x_6 = -\sqrt{2}, x_7 = \sqrt{2}$

r) $x_1 = -2, x_2 = 1, x_3 = 3, x_4 = 9, x_5 = -\sqrt{3}, x_6 = \sqrt{3}$

s) $x_1 = -25, x_2 = -5, x_3 = -1, x_4 = 2$

t) $x_1 = -3, x_2 = -2, x_3 = -2, x_4 = 1, x_5 = 1, x_6 = 3$

u) $x_1 = -4, x_2 = -3, x_3 = 3, x_4 = 4$

v) $x_1 = -5, x_2 = -2, x_3 = 2, x_4 = 5$

w) $x_1 = -5, x_2 = 5$

x) $x_1 = -6, x_2 = 6$

y) $x_1 = -6, x_2 = 5, x_3 = 7$

z) $x_1 = -7, x_2 = 4, x_3 = 6$

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